Skew Mal'cev-Neumann Series Ring Over a Dedekind Domain

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Factorization of one-sided ideals

One-sided ideals factor in an ”essentially unique” way into a product of ”atoms”.

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Non-examples

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- the matrix polynomial ring $M_n(D)[x]$ over a division ring is not a domain

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If we define $\langle I \rangle + \langle J \rangle := \langle K \rangle$ iff $\langle I \oplus J \rangle = \langle R \oplus K \rangle$, then

$$\mathcal{C}(R) = \{\langle I \rangle \mid I \text{ right } R\text{-ideal}\}$$

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is a group called the Steinitz class group. As in the commutative case, $K_0(R) \cong \mathbb{Z} \oplus \mathcal{C}(R)$.

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If we define the addition by components and multiplication respecting the multiplication in D and G together with the relation

$$ga = \alpha(g)(a)g,$$

the set R becomes a ring called *Mal'cev-Neumann series ring*.

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What is the Steinitz class group of R?

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There is an morphism

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Do the above elements generate the kernel?

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Do the above elements generate the kernel?

For $R = D[x;\sigma][x^{-1}]$ the answer is positive. Of course in this case, we consider the map $K_0(D) \to K_0(R)$.

References

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