Joint Meeting AMS UMI, Palermo 2024

Star operations related to polynomial closure

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July 23$^{\text{th}}$, 2024

Let D be an integral domain with quotient field K.

For each subset $E \subseteq K$, $\mathrm{Int}(E, D) := \{f \in K[X] \mid f(E) \subseteq D\}$ is called the *ring of D-integer-valued polynomials over E*. When $E = D$ we set $\mathrm{Int}(D) := \mathrm{Int}(D, D)$.

The *polynomial closure* (in D) of E is defined as the set

$$\mathrm{cl}_D(E) := \{x \in K \mid f(x) \in D, \forall\, f \in \mathrm{Int}(E, D)\},$$

$\mathrm{cl}_D(E)$ is the largest subset of K such that

$$\mathrm{Int}(E, D) = \mathrm{Int}(\mathrm{cl}_D(E), D)$$

If $E, F \subseteq K$ are such that $\mathrm{Int}(E, D) = \mathrm{Int}(F, D)$ we say that they are polynomially equivalent.

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A subset $E \subseteq K$ such that $\mathrm{Int}(E, D) = \mathrm{Int}(D)$ is said polynomially dense in D.

For instance $\mathbb{Z}^+$, $\mathbb{Z}^-$ or the union of two coprime ideals in $\mathbb{Z}$ are polynomially dense in $\mathbb{Z}$.

The concept of polynomial closure was introduced by Gilmer (1989) and McQuillan (1991) in a topological context. For instance in a Dedekind domain with finite residue fields the polynomial closure of a subset E is the intersection of its topological closures in every maximal ideal-adic topology.

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Let $\mathfrak{F}(D)$ be the set of nonzero fractional ideals of D. A star-operation is a map $*: \mathfrak{F}(D) \longrightarrow \mathfrak{F}(D)$, $I \mapsto I^*$ satisfying the following properties for each $a \in K \backslash \{0\}$ and $I, J \in \mathfrak{F}(D)$:

($*1$) $(aD)^* = aD$; $(aI)^* = aI^*$;

($*2$) $I \subseteq I^*$;

($*3$) $I \subseteq J \Rightarrow I^* \subseteq J^*$;

($*4$) $(I^*)^* = I^*$.

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The divisorial closure or v-closure is the map:

$$I \mapsto I_v := (I^{-1})^{-1},$$

where $I^{-1} := \{x \in K \mid xI \subseteq D\}$. We have that $I^* \subseteq I_v$ for each $I \in \mathfrak{F}(D)$ and each star-operation $*$.

The identity is also star operation.

The set $\mathrm{Star}(D)$ of star operations on D has a natural order given by $\star_1 \leq \star_2$ if $I^{\star_1} \subseteq I^{\star_2}$, for every fractional ideal I. Under this order, $\mathrm{Star}(D)$ is a complete lattice whose minimum is the identity and whose maximum is the *v-operation* .

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some considerations about the v-operation

- Every D-homomorphism from I to D can be uniquely extended to a D-homomorphism from K to K $\Rightarrow$ identify $\mathrm{Hom}_D(I,D)$ with the subset of $\mathrm{Hom}_D(K,K)$ mapping I into D
- $I^{-1} \overset{\varphi}{\cong} \mathrm{Hom}_D(I,D)$ defined by $\varphi(a)(x) = ax$ for all $a \in I^{-1}$ and $x \in I$ (thus $\mathrm{Hom}_D(I,D) \cong I^{-1}X$)
- $I_v \overset{\lambda}{\cong} \mathrm{Hom}_D(\mathrm{Hom}_D(I,D),D)$ defined by $\lambda(x)(f) = f(x)$ for all $x \in I_v$ and $f \in \mathrm{Hom}_D(I,D)$
- $I_v = \{x \in K \mid f(x) \in D, \forall\ f \in \mathrm{Hom}_D(I,D)\}$
- I_v is the biggest ideal such that $\mathrm{Hom}_D(I,D) = \mathrm{Hom}(I_v,D)$.

The inclusion $I^{-1}X \subseteq \mathrm{Int}(I,D)$ implies that $I_v \supseteq \mathrm{cl}_D(I)$.

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Represent $\mathrm{Int}(I, D)$ when $\mathrm{Int}(D) = D[X]$

If $\mathrm{Int}(D) = D[X]$ then $\mathrm{Int}(I, D) = \cap_{a \in I \setminus \{0\}} D[X/a]$ and so it is the graded ring:

$$\mathrm{Int}(I, D) = D \oplus \Big(\bigcap_{u \in I \setminus \{0\}} \frac{1}{u} D\Big) X \oplus \cdots \oplus \Big(\bigcap_{u \in I \setminus \{0\}} \frac{1}{u^n} D\Big) X^n \oplus \cdots =$$

We let $I(n)$ denote the D-module generated by the set $\{u^n \mid u \in I\}$ and so $\bigcap_{u \in I \setminus \{0\}} \frac{1}{u^n} D = (D : I(n)) = I(n)^{-1}$.

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$$\mathrm{Int}(I, D) \supseteq \bigoplus_{n\in\mathbb{N}} I(n)^{-1}X^n = D[X/I]$$

Theorem

Let D be a domain and suppose that $\mathrm{Int}(D) = D[X]$. If I is a nonzero fractional ideal of D, then

$$\mathrm{cl}_D(I) = \{z \in K \mid z^n \in I(n)_v, \forall n \geq 0\}.$$

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the operations $\star_n$, $\star_\infty$

For any integer $n > 0$ we can define the operation

$$\begin{aligned}\star_n\colon \mathfrak{F}(D) &\longrightarrow \mathfrak{F}(D),\\ I &\longmapsto \{x \in K \mid x^t \in I(t)_v \text{ for all } t \leq n\}\end{aligned}$$

and we can also define the operation

$$\begin{aligned}\star_\infty\colon \mathfrak{F}(D) &\longrightarrow \mathfrak{F}(D),\\ I &\longmapsto \bigcap\{I^{\star_n} \mid n \in \mathbb{N}\}\\ &\{x \in K \mid x^n \in I(n)_v \text{ for all } n \in \mathbb{N}\}.\end{aligned}$$

Proposition

The $\star_n$ and $\star_\infty$ are star operations on D, and

$$v = \star_1 \geq \star_2 \geq \star_3 \geq \cdots \geq \star_\infty$$

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Let $D[X/I]_n := \bigoplus_{t=0}^{n} I(t)^{-1}X^t$ be the set of polynomials of $D[X/I]$ of degree at most n.

Proposition

(a) For each $n \geq 1$, $I^{\star_n} := \{z \in K \mid f(z) \in D, \ \forall f \in D[X/I]_n\}$;

(b) $I^{\star_\infty} := \{z \in K \mid f(z) \in D, \ \forall f \in D[X/I]\}$.

Note that since $D[X/I] \subseteq \mathrm{Int}(I, D)$, then $\mathrm{cl}_D \leq \star_\infty$ and if $\mathrm{Int}(D) = D[X]$ then $\star_\infty = \mathrm{cl}_D$.

Then we have the chain:

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Corollary

$I^{\star_n}$ is the largest set such that $D[X/I]_n = D[X/I^{\star_n}]_n$ and $I^{\star_\infty}$ is the largest set such that $D[X/I] = D[X/I^{\star_\infty}]$.

The above result replicates the fact that the polynomial closure of E is the largest subset of K such that $\mathrm{Int}(E, D) = \mathrm{Int}(\mathrm{cl}_D(E), D)$ and I_v is the biggest ideal such that $\mathrm{Hom}_D(I, D) = \mathrm{Hom}(I_v, D)$.

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M.H. Park - F.T in 2005 proved that when D is an essential domain, then $\star_\infty = v$ and in many subcases of essential domains we have that $\mathrm{cl}_D = v$.

The key-tool to prove this equality is the fact that for any ideal I in an essential domain

$$(\diamond) \quad I(n)_v = (I^n)_v$$

In fact:

- $I^{\star_\infty} = I_v \Leftrightarrow I(n)_v = I_v(n)_v$ (for the maximality of the v-operation)
- it is known that $(I^n)_v = (I_v^n)_v$ for any ideal I in any domain D
- if $I(n)_v = (I^n)_v$, from the above conditions we have that $\star_\infty = v$.

We can obtain the condition $(\diamond)$ when D is integrally closed so generalizing the result obtained for essential domains.

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- $I^{\star_\infty} = I_v \Leftrightarrow I(n)_v = I_v(n)_v$ (for the maximality of the v-operation)
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We can obtain the condition $(\diamond)$ when D is integrally closed so generalizing the result obtained for essential domains.

M.H. Park - F.T in 2005 proved that when D is an essential domain, then $\star_\infty = v$ and in many subcases of essential domains we have that $\mathrm{cl}_D = v$.
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We can obtain the condition $(\diamond)$ when D is integrally closed so generalizing the result obtained for essential domains.

integrally closed domain

Theorem

Let D be an integrally closed domain. Then $\star_\infty = v$.

tools

- if $I(n)^\star = (I^n)^\star$ for a star operation $\star$ then $\star_\infty = v$. Indeed, for the maximality of the v-operation we have that $I(n)^\star = (I^n)^\star \Rightarrow I(n)_v = (I^n)_v$.
- we take the b-operation - $I^b = \bigcap\{IV \mid D \subseteq V \in K\}$ - and show that $I(n)^b = (I^n)^b$ for every ideal I and integer n (it is enough to show that $I(n)V = I^nV$ for all valuation overrings V).

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Consider the case $\mathbb{Q} \subset D$. This is equivalent to ask that the residue fields are of characteristic 0 and so $\mathrm{Int}(D) = D[X]$. Claim

- $I(n) = I^n$

Lemma W

Let X, Y be indeterminates over $\mathbb{Q}$. For every n, the sets $\{X^n, (X+1)^n, (X+2)^n, \ldots, (X+n)^n\}$ and $\{X^n, (X+Y)^n, (X+2Y)^n, \ldots, (X+nY)^n\}$ are linearly independent over $\mathbb{Q}$.

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Proposition

Let D be an integral domain with $\mathbb{Q} \subseteq D$ and $I \in \mathfrak{F}(D)$. Then, $I(n) = I^n$ for all $n \geq 0$.

Sketch of the proof It is enough to show that $I \cdot I(n-1) = I(n)$ for all $n \geq 1$: indeed, if this equality is true, then $I(n) = I \cdot I(n-1) = I^2 \cdot I(n-2) = \cdots = I^{n-1} \cdot I(1) = I^{n-1} \cdot I = I^n$.
The containment $I(n) \subseteq I \cdot I(n-1)$ is obvious.
For the reverse containment, $I \cdot I(n-1) = \langle xy^{n-1}, x, y \in I \rangle$ and we show that the elements xy^{n-1} are in $I(n)$ by dimension considerations based on Lemma W.

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The Proposition above does not hold in general for rings of characteristic 0 not containing $\mathbb{Q}$. For example, if $D = \mathbb{Z}[X, Y]$ and $I = (X, Y)$, then $XY \in I^2$ but $XY \notin I(2)$.

Theorem

Let D be a domain such that $D/\mathfrak{m}$ has characteristic 0, for each maximal ideal $\mathfrak{m}$. Then $\star_\infty = \mathrm{cl}_D = v$.

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caharacteristic p

$$v = \star_1 \geq \star_2 \geq \star_3 \geq \cdots \geq \star_\infty \geq \mathrm{cl}_D$$

In characteristic p it is not always true that all the $\star_n$ are equal.

Example Let $F \subseteq K \subseteq L$ be a tower of purely inseparable extension of degree p, with $L = F(y)$ simple over F. Consider

$$D := F + XL[[X]], \quad I := K + XL[[X]]$$

then, $I^{\star_1} = I_v = L[[X]]$. On the other hand, $I(p) = K(p) + XL[[X]] = K^p + XL[[X]] = D$, and thus $I(p)_v = D$; therefore, $y^p \notin I(p)$ since $y^p \notin F$. It follows that $I^{\star_p} \neq L[[X]]$ and thus $\star_p \neq \star_1$.

The main difference from the previous case is that Lemma W does not hold (the determinant of the Wronskian matrix in the proof of the Lemma must be nonzero - it may be equal to a multiple of p).

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Proposition

Let D be a ring of characteristic p containing an infinite field. Let $n = t_0 + t_1 p + \cdots + t_k p^k$, with $0 \leq t_i < p$ for every i. Then,

$$I(n) = I^{t_0} \cdot I(p)^{t_1} \cdot I(p^2)^{t_2} \cdots I(p^k)^{t_k}.$$

Corollary

Let D be a ring of characteristic p containing an infinite field. Then $\mathrm{cl}_D(I) = I^{\star\infty} = \{x \in K \mid x^{p^e} \in I(p^e)_v \text{ for every } e \geq 0\}$.

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If D has characteristic p we can put some extra hypothesis to get that $\mathrm{cl}_D = \star_\infty = v$:

(1) D contains an infinite field;

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(1) implies that $\mathrm{cl}_D = \star_\infty$ and that $I^{\star_\infty} = \{x \in K \mid x^{p^e} \in I(p^e)_v$ for every $e \geq 0\}$;

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Theorem

Let D be an integral domain of characteristic p that contains an infinite field and such that every element of D has a p-th root in D. Then, $\mathrm{cl}_D = v$.

Example Let F be a perfect infinite field, and let L be an algebraic extension of F. Consider the ring

$$D := \bigcup_{n\geq 1} (F + X^{1/p^n} L[[X^{1/p^n}]])$$

D contains an infinite field (F) and every element has a p-th root.

Indeed, if $x \in D$ then $x \in F + X^{1/p^n} L[[X^{1/p^n}]]$ for some n and we can write $x = \sum_{i\geq 0} a_i X^{i/p^n}$ with $a_0 \in F$ and $a_i \in L$ for all $i > 0$. Since both F and L are perfect, there are $b_0 \in F$ and $b_i \in L$ (for $i > 0$) such that $b_j^p = a_j$ for all j. Setting $y := \sum b_i X^{i/p^{n+1}}$, then $y \in D$ and $y^p = x$.

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Thank you!