Differences in sets of lengths for monoids of plus-minus weighted zero-sum sequences

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Outline

Zero-sum sequences

Distances and differences

Weighted zero-sum sequences

Results for (plus-minus) weighted zero-sum sequences

Characterization of groups by sets of lengths

The monoid of zero-sum sequences, aka the block monoid, $\mathcal{B}(G_0)$

Let $(G, +, 0)$ be a (finite) abelian group. Let $G_0 \subset G$. A *sequence* S over G_0 is an element of $\mathcal{F}(G_0)$ the free abelian monoid with basis G_0.

Thus a sequences is a (formal, commutative) product

$$S = \prod_{i=1}^{l} g_i = \prod_{g \in G_0} g^{\mathsf{v}_g(S)}.$$

The sequence S is called a *zero-sum sequence* if its *sum*

$$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G_0} \mathsf{v}_g(S) g \in G$$

equals 0.

The monoid of zero-sum sequences over G_0 is defined as

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \colon \sigma(S) = 0\}.$$

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Study the arithmetic: sets of lengths

A monoid H (commutative, cancellative), for example the multiplicative monoid of a domain, is called *atomic* if each non-zero element a is the product (of finitely many) irreducible elements.

If

$$a = a_1 \dots a_n$$

with irreducible a_i, then n is called a length of a.

$$\mathsf{L}(a) = \{n \colon n \text{ is a length }\}.$$

For a invertible set $\mathsf{L}(a) = \{0\}$.
The *system of sets of lengths* is

$$\mathcal{L}(H) = \{\mathsf{L}(a) \colon a \in H\}.$$

If all sets of lengths are singletons, the structure is called half-factorial (Zaks, 1976).

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Applications of monoids of zero-sum sequences

Various monoids and domains of interest admit a transfer-homomorphism to monoids of zero-sum sequences (or other auxiliary monoids). They preserve sets of lengths.

For a Krull monoid H sets of lengths just depend on the class group $\mathcal{C}(H) = G$ and the set G_0 of classes containing primes (the distribution of prime v-ideals).

More precisely, there exists a transfer homorphism (the block homomorphism)

$$\beta : H \to \mathcal{B}(G_0)$$

such that

$$\mathsf{L}_H(a) = \mathsf{L}_{\mathcal{B}(G_0)}(\beta(a))$$

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Sets of distances

For $A \subseteq \mathbb{Z}$, we denote by $\Delta(A)$ the set of (successive) distances of A, that is the set of all $d \in \mathbb{N}$ for which there exists $\ell \in A$ such that $A \cap [\ell, \ell + d] = \{\ell, \ell + d\}$. Clearly, $\Delta(A) \subseteq \{d\}$ if and only if A is an arithmetical progression with difference d.

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Sets of differences, or minimal distances

Let $\Delta^*(H) = \{\min \Delta(H') \colon H' \subset H$ divisor-closed, and not HF$\}$ the set of minmal distances.
Introduced by Gao and Geroldinger (2000).
A main motivation is that in case the *Structure Theorem of Sets of Lengths* holds, that is sets of lengths are almost arithmetical multiprogressions with globally bounded parameters, this set (usually) gives the sets of differences for these arithmetic progressions.
There is some $M \in \mathbb{N}_0$ such that each set of lengths L of H is an almost arithmetical multiprogression with bound M and *difference* $d \in \Delta^*(H) \cup \{0\}$, that is,

$$L = y + (L_1 \cup L^* \cup (\max L^* + L_2)) \subseteq y + \mathcal{D} + d\mathbb{Z}$$

with $y \in \mathbb{N}_0$, $\{0, d\} \subseteq \mathcal{D} \subseteq [0, d]$, $-L_1, L_2 \subseteq [1, M]$, $\min L^* = 0$ and $L^* = [0, \max L^*] \cap \mathcal{D} + d\mathbb{Z}$.

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A fundamental lemma It is known that $\min \Delta(H) = \gcd \Delta(H)$. (Geroldinger)

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Weighted zero-sum sequences

Let $(G, +, 0)$ be a (finite) abelian group. Let $G_0 \subset G$. Let Ω be "a set of weights." Let $S = \prod_{i=1}^{l} g_i$ be a sequence.
Then any elements of the form

$$\sum_{i=1}^{l} \omega_i g_i$$

with $\omega_i \in \Omega$ is called an Ω-weighted sum of S.
What do we take as set of weights?

1. Subset of the integers, or of $\{0, 1, \ldots, \exp(G) - 1\}$.
2. Subset of the endomorphisms of $\mathrm{End}(G)$ (more general).

Let $\sigma_\Omega(S)$ denote the set of all elements that are an Ω-weighted sum of S.
We say that S is a Ω-weighted zero-sum sequence if $0 \in \sigma_\Omega(S)$.
Note: The sequences is not 'weighted', the sum is.

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$$\mathcal{B}_\Omega(G) = \{S \in \mathcal{F}(G) \colon 0 \in \sigma_\Omega(S)\} \subset \mathcal{F}(G)$$

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$\mathcal{B}_\Omega(G)$ is a submonoid of $\mathcal{F}(G)$.

Moreover $\mathcal{B}(G) \subset \mathcal{B}_\Omega(G)$.

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Motivation

Let K denote a Galois number field. Let $\mathcal{O}_K$ denote its ring of algebraic integers.
Let $\mathrm{N} : \mathcal{O}_K^* \to \mathbb{N}$ denote the absolute norm.
Then $\mathrm{N}(\mathcal{O}_K^*)$ is a submonoid of $(\mathbb{N}^*, \cdot)$. We want to study the arithmetic of that monoid.

Theorem (Boukheche, Merito, Ordaz, S.)

Let K be a Galois number field with Galois group Γ and class group G. There is a transfer homomorphism from $\mathrm{N}(\mathcal{O}_K^*)$, the monoid of absolute norms of non-zero algebraic integers of K, to $\mathcal{B}_\Gamma(G)$, the monoid of Γ-weighted zero-sum sequences over the class group of K.

Recall that the Galois group acts on the class group; thus it makes sense to talk about Γ-weighted zero-sum sequences over the class group of K.
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Motivation II

Weighted zero-sum sequences are linked to coding theory. See talks in the “Algebraic coding theory” section (tomorrow afternoon)

- “The geometry of intersecting codes: bounds and constructions” (by M. Borello)
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The (ir-)reducible elements of $\mathcal{B}_\Omega(G)$

A non-empty/non-invertible $S \in \mathcal{B}_\Omega(G)$ is reducible if there are two non-empty elements $S_1, S_2 \in \mathcal{B}_\Omega(G)$ such that $S = S_1 S_2$. That is, S can be decomposed into two non-empty Ω-weighted zero-sum sequences S_1 and S_2.

That is, $S = S_1 S_2$ with $0 \in \sigma_\Omega(S_1)$ and $0 \in \sigma_\Omega(S_2)$.

Note: Contrary to the case without weights, it does not suffice that there exist some proper divisor S_1 of S with $0 \in \sigma_\Omega(S_1)$, because $0 + a = 0$ implies $a = 0$, but $0 \in A_1$ and $0 \in A_1 + A_2$ does not imply $0 \in A_2$.

We denote by $\mathcal{A}(\mathcal{B}_\Omega(G))$ the set of irreducible Ω-weighted zero-sum sequences.

These monoids are usually not Krull, but are C-monoids. (A submonoid of a free monoids is a C-monoid if its [reduced] class semigroup is finite.)

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That is, S can be decomposed into two non-empty Ω-weighted zero-sum sequences S_1 and S_2.
That is, $S = S_1 S_2$ with $0 \in \sigma_\Omega(S_1)$ and $0 \in \sigma_\Omega(S_2)$.
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We denote by $\mathcal{A}(\mathcal{B}_\Omega(G))$ the set of irreducible Ω-weighted zero-sum sequences.
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It is not hard to see that $\mathcal{B}_\Omega(G)$ is finitely generated.
This has immediate arithmetic consequences. In particular the Structure Theorem for Sets of Lengths holds, this means that the sets of lengths are almost arithmetical multiprogressions with globaly bounded parameters.
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What can we say about $\Delta^*(\mathcal{B}_{\pm}(G))$, that is the set of $\min \Delta(H)$ for divisor-closed submonoids of $\mathcal{B}_{\pm}(G)$?

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A result for groups of odd order

Theorem (Merito, Ordaz, S.)

If $|G|$ *odd then* $\max \Delta^*(\mathcal{B}_\pm(G)) = \exp(G) - 2$.

For comparison $\max \Delta^*(\mathcal{B}(G)) = \max\{\exp(G) - 2, \mathrm{r}(G) - 1\}$ (Geroldinger, Zhong), but that's much harder to prove.

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Lemma

Let $A \in \mathcal{A}(\mathcal{B}_{\pm}(G))$ *and* $A \neq 0$*. Then* $\{2, |A|\} \subset \mathsf{L}(A^2)$*.*

Proof: Let $A = g_1 \dots g_k$. Then

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is a factorization as $0 = (+1)g_i + (-1)g_i$.

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Assume that the order of g is odd, then $g^{\mathrm{ord}(g)} \in \mathcal{A}(\mathcal{B}_{\pm}(G))$.

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Somewhat stronger version of the result

Theorem

Let G be a finite abelian group exponent n and let $H = \mathcal{B}_{\pm}(G)$. Assume that $n \geq 3$ is odd. Let $D_1 = \{d - 2 \colon d \mid n, d \geq 3\}$ and let $D_2 = \{d' \mid d \colon d \in D_1\}$. Then $D_1 \subseteq \Delta^*(H) \subseteq D_2$. In particular, $\max \Delta^*(H) = n - 2$.

A consequence

Corollary

Let p be a prime such that $p-2$ is prime. Then for $G = C_p^r$ one has $\Delta^*(\mathcal{B}_\pm(G)) = \{1, p-2\}$. In particular, for $p = 3$ one has $\Delta^*(\mathcal{B}_\pm(G)) = \{1\}$.

Note: This also holds for elementary *p*-groups of infinite rank.
Note without weights $\Delta^*(\mathcal{B}(G)) = \mathbb{N}$ for infinite G (Chapman, S., Smith) while $\Delta^*(\mathcal{B}_\pm(G)) = \{1\}$ for $G = C_3^{(\mathbb{N})}$
Nevertheless, by a result of Geroldinger and Kainrath we know that 'every' set is a set of length for $\mathcal{B}_\pm(G)$ for infinite G.

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What about the case of even exponent?

Basic construction

Lemma

Let G be a finite abelian group and let $e_1, \ldots, e_r$ be independent elements of even order, say $\operatorname{ord}(e_i) = 2m_i$. Assume that $m_1 + \cdots + m_r \geq 2$. Let $e_0 = m_1 e_1 + \cdots + m_r e_r$, $G_0 = \{e_0, e_1, \ldots, e_r\}$ and $H = \mathcal{B}_{\pm}(G_0)$. Then $\Delta(H) = \{m_1 + \cdots + m_r - 1\}$.

Proof: $A = e_0 e_1^{m_1} \ldots e_r^{m_r}$ is an atom. The only other atoms are e_i^2. So $\mathsf{L}(A^2) = \{2, |A|\}$.
Note that $m_1 + \cdots + m_r - 1$ can significantly exceed $\exp(G) - 2$ and $\mathsf{r}(G)$.

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Curious link

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This is a standard lower bound for $\max \Delta(\mathcal{B}(G))$. In various cases this should actually be $\max \Delta(\mathcal{B}(G))$ (possibly often/always, unless $\exp(G) - 2$ is larger).

Characterization of groups by sets of lengths

Let G_1 and G_2 be finite abelian groups. Suppose that $\mathcal{L}(\mathcal{B}_{\pm}(G_1)) = \mathcal{L}(\mathcal{B}_{\pm}(G_2))$. Is it true that G_1 and G_2 are isomorphic?

This problem was studied a lot without weights.

Obvioulsy C_1 and C_2 is a counter example.

Moreover, C_3, C_4, C_2^2 is also a counter-example as they all give the same system of sets of lengths for $\mathcal{B}_{\pm}(G)$ (by a result of Fabsits, Geroldinger, Reinhart, Zhong). Specifically, the system of sets of lengths is $\{y + 2k + [0, k] \colon y, k \in \mathbb{N}_0\}$.

However, they showed if $\mathcal{L}(\mathcal{B}_{\pm}(G)) = \mathcal{L}(\mathcal{B}_{\pm}(C_n))$ for $n \geq 5$ then indeed $G \simeq C_5$.

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Our resuls on $\Delta^* H$ imply, for G a finite abelian group and $H = \mathcal{B}_{\pm}(G)$, one has $\max \Delta^*(H) = 1$ if and only if $\exp(G) = 3$ or $G = C_2^2$ or $G = C_4$.

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Differences in sets of lengths for monoids of plus-minus weighted zero-sum sequences

Wolfgang Schmid, joint with Kamil Merito and Oscar Ordaz

LAGA, Université Paris 8

July, 2024
Palermo