On the arithmetic of the monoid of plus-minus weighted zero-sum sequences

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July 24, 2024

This talk is based on a joint work with Florin Fabsits, Alfred Geroldinger and Qinghai Zhong

NAWI Graz Natural Sciences

FWF Der Wissenschaftsfonds.

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Set $H_{\mathrm{red}} = H/H^\times$, set $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\mathrm{red}}))$, and let $\pi : \mathsf{Z}(H) \to H_{\mathrm{red}}$ be the **factorization homomorphism**.

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For each $z \in \mathsf{Z}(H)$, let $|z|$ be the **length** of z.

Various types of closures

Let $H' = \{z \in K : \text{there is } n \in \mathbb{N} \text{ such that } z^k \in H \text{ for each } k \in \mathbb{N}_{\geq n}\}$, called the **seminormal closure** of H.

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Observe that $H' \subseteq \widetilde{H} \subseteq \widehat{H}$.
Also, H is called **seminormal** (**root closed**, **completely integrally closed**) if $H = H'$ ($H = \widetilde{H}$, $H = \widehat{H}$).

Finitely generated/Mori/Krull monoids

For each $X \subseteq K$, set $(H : X) = \{z \in K : zX \subseteq H\}$ and $X_v = (H : (H : X))$, called the v-**closure** of X. If $X \subseteq H$ is such that $X_v = X$, then X is called a **divisorial ideal** of H.

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If H is a finitely generated monoid, then H is a Mori monoid, $(H : \widehat{H}) \neq \emptyset$ and $\widehat{H} = \widetilde{H}$ is a Krull monoid.
Let $\mathcal{C}_v(H)$ denote the **divisor class group** of H.

Sets of lengths and more

For each $a \in H$ set $\mathsf{Z}(a) = \pi^{-1}(aH^\times)$, $\mathsf{L}(a) = \{|z| : z \in \mathsf{Z}(a)\}$ and $\Delta(a) = \{\ell - k : \ell, k \in \mathsf{L}(a), k < \ell, \mathsf{L}(a) \cap [k, \ell] = \{k, \ell\}\}$.

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We set $\Delta(H) = \bigcup_{a \in H} \Delta(a)$, called the **set of distances** of H.
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We say that H is **(half-)factorial** if $\mathsf{Z}(a)$ (resp. $\mathsf{L}(a)$) is a singleton for each $a \in H$.

Furthermore, H is said to be a **BF-monoid** if $\mathsf{L}(a)$ is finite and nonempty for each $a \in H$ and H is called an **FF-monoid** if $\mathsf{Z}(a)$ is finite and nonempty for each $a \in H$.

C-monoids

Let M be a monoid and let $H \subseteq M$ be a submonoid. Two elements $x, y \in M$ are called (H, M)-equivalent (denoted by $x \sim y$) if $x^{-1}H \cap M = y^{-1}H \cap M$.

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Note that $\sim$ is an equivalence relation on M.
For each $x \in M$, let $[x]_\sim$ be the equivalence class of x.
Set $\mathcal{C}^*(H, M) = \{[z]_\sim : z \in (M \setminus M^\times) \cup \{1\}\}$.

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Let F be a factorial monoid and let $H \subseteq F$ be a submonoid. We say that H is a **C-monoid defined in** F if $F^\times \cap H = H^\times$ and $\mathcal{C}^*(H, F)$ is finite. Moreover, H is a **C-monoid** if it is a C-monoid defined in some factorial monoid.

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If H is a C-monoid, then H is a Mori monoid, $(H : \widehat{H}) \neq \emptyset$, $\widehat{H}$ is a Krull monoid and $\mathcal{C}_v(\widehat{H})$ is finite.

Some homomorphisms

Let B be a monoid and let $\varphi : H \to B$ be a monoid homomorphism.

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We say that $\varphi : H \to B$ is a **transfer homomorphism** if $B = \varphi(H)B^\times$, $\varphi^{-1}(B^\times) = H^\times$ and for all $a \in H$ and $b, c \in B$ with $\varphi(a) = bc$, there are some $b', c' \in H$ and $\beta, \gamma \in B^\times$ such that $b = \varphi(b')\beta$, $c = \varphi(c')\gamma$ and $a = b'c'$.

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Now let B be a free abelian monoid with basis P and let φ be a divisor homomorphism. Then φ is called a **divisor theory** if for each $p \in P$, there is some finite $E \subseteq H$ such that $p = \gcd(\varphi(E))$.

Transfer Krull monoids

Note that H is a Krull monoid if and only if there exists a factorial monoid F and a divisor homomorphism $\varphi : H \to F$.
Moreover, H is a Krull monoid if and only if there is a free abelian monoid B and a divisor theory $\psi : H \to B$.

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We say that H is **transfer Krull** if there exists a Krull monoid M and a transfer homomorphism $\psi : H \to M$.
Clearly, every Krull monoid is transfer Krull. Also note that every half-factorial monoid is transfer Krull.

Monoids of (weighted) zero-sum sequences

Let G be an abelian group, let $\Gamma \subseteq \mathrm{End}(G)$ and let $\mathcal{F}(G)$ be the free abelian monoid with basis G.
Set $\mathcal{B}_\Gamma(G) = \{\prod_{i=1}^n x_i \in \mathcal{F}(G) : \sum_{i=1}^n \gamma_i(x_i) = 0$ for some $(\gamma_i)_{i=1}^n \in \Gamma^{[1,n]}\}$, called the **monoid of Γ-weighted zero-sum sequences** over G.

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Observe that $\mathcal{B}_\Gamma(G)$ is both an FF-monoid and a BF-monoid.

Set $\mathcal{B}(G) = \mathcal{B}_{\{\mathrm{id}\}}(G)$, called the **monoid of zero-sum sequences** over G and set $\mathcal{B}_\pm(G) = \mathcal{B}_{\{\mathrm{id},-\mathrm{id}\}}(G)$, called the **monoid of plus-minus weighted zero-sum sequences** over G.

A classical result

Theorem

Let K be an algebraic number field, let $\mathcal{O}_K$ be the ring of algebraic integers in K, let G be the class group of $\mathcal{O}_K$ and let $M = \mathcal{O}_K \setminus \{0\}$ be the monoid of nonzero elements of $\mathcal{O}_K$. Then there exists a transfer homomorphism $\theta : M \to \mathcal{B}(G)$.

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Next we want to point out that a similar type of connection can be established involving monoids of weighted zero-sum sequences.

An important motivation

Theorem (Boukheche, Merito, Ordaz, Schmid, 2022)

Let $K/\mathbb{Q}$ be a finite Galois extension, let $\mathcal{O}_K$ be the ring of algebraic integers in K, let G be the class group of $\mathcal{O}_K$ and let Γ be the Galois group of $K/\mathbb{Q}$. Let $\mathrm{N} : K \to \mathbb{Q}$ be the norm map and let $\mathrm{N}(\mathcal{H}_K) = \{|\mathrm{N}(a)| : a \in \mathcal{O}_K \setminus \{0\}\}$ be the norm monoid. Then there exists a transfer homomorphism $\theta : \mathrm{N}(\mathcal{H}_K) \to \mathcal{B}_\Gamma(G)$.

Elementary results

Proposition (Fabsits, Geroldinger, R., Zhong, 2024)

Let G be an abelian group.

(1) If $|G| \le 2$, then

$$\mathcal{B}(G) = \mathcal{B}_{\pm}(G) \cong \mathcal{F}(G) \cong (\mathbb{N}_0^{|G|}, +) .$$

(2) The following statements are equivalent.

  (a) $|G| \le 2$.
  (b) $\mathcal{B}_{\pm}(G)$ is factorial.
  (c) $\mathcal{B}_{\pm}(G)$ is half-factorial.

On the root closure

Proposition (Fabsits, Geroldinger, R., Zhong, 2024)

Let G be an abelian group.

(1) $\widetilde{\mathcal{B}_{\pm}(G)} = \widehat{\mathcal{B}_{\pm}(G)}$ is a Krull monoid.

(2) If $|G| \neq 2$, then the inclusion $\widetilde{\mathcal{B}_{\pm}(G)} \hookrightarrow \mathcal{F}(G)$ is a divisor theory.

On the Mori property

Theorem (Fabsits, Geroldinger, R., Zhong, 2024)

Let G be an abelian group. Then the following statements are equivalent.

(a) $\mathcal{B}_{\pm}(G)$ is a Mori monoid.

(b) $(\mathcal{B}_{\pm}(G) : \widehat{\mathcal{B}_{\pm}(G)}) \neq \emptyset$.

(c) $2G$ is finite.

(d) $G = G_1 \oplus G_2$, where G_1 is an elementary 2-group and G_2 is a finite group.

If these equivalent conditions are satisfied, then $\mathcal{B}_{\pm}(G)$ is seminormal if and only if $\exp(G) \mid 4$.

Rediscovering some facts

Corollary (Fabsits, Geroldinger, R., Zhong, 2024)

Let G be an abelian group. Then the following statements are equivalent.

(a) $\mathcal{B}_{\pm}(G)$ is a Krull monoid.

(b) $\mathcal{B}_{\pm}(G)$ is completely integrally closed.

(c) $\mathcal{B}_{\pm}(G)$ is root closed.

(d) $\mathcal{B}_{\pm}(G)$ is a transfer Krull monoid.

(e) G is an elementary 2-group.

On C-monoids and finitely generated monoids

Theorem (Fabsits, Geroldinger, R., Zhong, 2024)

Let G be an abelian group. Then the following statements are equivalent.

(a) $\mathcal{B}_{\pm}(G)$ is finitely generated.

(b) $\mathcal{B}_{\pm}(G)$ is a C-monoid defined in $\mathcal{F}(G)$.

(c) $\mathcal{B}_{\pm}(G)$ is a C-monoid.

(d) $\mathcal{B}_{\pm}(G)$ is a Mori monoid and $\mathcal{C}_v\big(\widehat{\mathcal{B}_{\pm}(G)}\big)$ is finitely generated.

(e) G is finite.

The isomorphism problem I

Theorem

Let G_1 and G_2 be abelian groups. Then $\mathcal{B}(G_1)$ and $\mathcal{B}(G_2)$ are isomorphic if and only if G_1 and G_2 are isomorphic.

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This motivates the **isomorphism problem for plus-minus weighted zero-sum sequences**:
Let $\mathcal{B}_\pm(G_1)$ and $\mathcal{B}_\pm(G_2)$ be isomorphic. Under what circumstances are G_1 and G_2 isomorphic?

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The isomorphism problem has been studied in various other situations. For instance, it was investigated for monoids of product-one sequences (e.g. by Geroldinger and Oh) and for power monoids (e.g. by Tringali and Yan).

The isomorphism problem II

Proposition (Fabsits, Geroldinger, R., Zhong, 2024)

Let G_1 and G_2 be abelian groups such that $|G_1|, |G_2| \neq 2$ and let $\varphi : \mathcal{B}_{\pm}(G_1) \to \mathcal{B}_{\pm}(G_2)$ be a monoid isomorphism.

(1) $\varphi(0) = 0$ and $|A| = |\varphi(A)|$ for every $A \in \mathcal{B}_{\pm}(G_1)$.

(2) For every $g \in G_1$, there exists $h \in G_2$ with $\mathrm{ord}(h) = \mathrm{ord}(g)$ such that $\varphi(g^2) = h^2$.

(3) For every $h \in G_2$, there exists $g \in G_1$ such that $\varphi(g^2) = h^2$.

(4) There is a bijection $\varphi_0 : G_1 \to G_2$.

The isomorphism problem III

Theorem (Fabsits, Geroldinger, R., Zhong, 2024)

Let G_1 and G_2 be abelian groups and suppose that G_1 is a direct sum of cyclic groups. Then the groups G_1 and G_2 are isomorphic if and only their monoids of plus-minus weighted zero-sum sequences $\mathcal{B}_{\pm}(G_1)$ and $\mathcal{B}_{\pm}(G_2)$ are isomorphic.

Direct sums of cyclic groups

Remark

Let G be an abelian group.

(1) There are important situations in which G is a direct sum of cyclic groups. This is the case if G is free abelian, or if G is finitely generated, or if G is bounded (i.e., $\exp(G) < \infty$).

(2) Nevertheless, G need not always be a direct sum of cyclic groups. For instance, if $G = \mathbb{Q}/\mathbb{Z}$, then G is a torsion group, but not a direct sum of cyclic groups.

The characterization problem I

Let G_1 and G_2 be abelian groups. Clearly, if G_1 and G_2 are isomorphic, then $\mathcal{L}(\mathcal{B}_{\pm}(G_1)) = \mathcal{L}(\mathcal{B}_{\pm}(G_2))$.

The characterization problem I

Let G_1 and G_2 be abelian groups. Clearly, if G_1 and G_2 are isomorphic, then $\mathcal{L}(\mathcal{B}_{\pm}(G_1)) = \mathcal{L}(\mathcal{B}_{\pm}(G_2))$.

This leads to the **characterization problem for monoids of plus-minus weighted zero-sum sequences**:
Suppose that $\mathcal{L}(\mathcal{B}_{\pm}(G_1)) = \mathcal{L}(\mathcal{B}_{\pm}(G_2))$. Under what circumstances are G_1 and G_2 isomorphic?

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Let G_1 and G_2 be abelian groups. Clearly, if G_1 and G_2 are isomorphic, then $\mathcal{L}(\mathcal{B}_{\pm}(G_1)) = \mathcal{L}(\mathcal{B}_{\pm}(G_2))$.

This leads to the **characterization problem for monoids of plus-minus weighted zero-sum sequences**:
Suppose that $\mathcal{L}(\mathcal{B}_{\pm}(G_1)) = \mathcal{L}(\mathcal{B}_{\pm}(G_2))$. Under what circumstances are G_1 and G_2 isomorphic?

Note that if G_1 and G_2 are infinite abelian groups, then it follows from results of Geroldinger and Kainrath that $\mathcal{L}(\mathcal{B}_{\pm}(G_1)) = \mathcal{L}(\mathcal{B}_{\pm}(G_2))$.

The characterization problem II

Theorem (Fabsits, Geroldinger, R., Zhong, 2024)

(1) $\mathcal{L}\big(\mathcal{B}_{\pm}(C_1)\big) = \mathcal{L}\big(\mathcal{B}_{\pm}(C_2)\big) = \big\{\{k\} : k \in \mathbb{N}_0\big\}$.

(2) $\mathcal{L}\big(\mathcal{B}_{\pm}(C_3)\big) = \mathcal{L}\big(\mathcal{B}_{\pm}(C_4)\big) = \mathcal{L}\big(\mathcal{B}_{\pm}(C_2 \oplus C_2)\big) = \big\{y + 2k + [0, k] : y, k \in \mathbb{N}_0\big\}$.

(3) $\mathcal{L}\big(\mathcal{B}_{\pm}(C_2^3)\big) = \big\{y + (k+1) + [0, k] : y \in \mathbb{N}_0, k \in [0, 2]\big\} \cup \big\{y + k + [0, k] : y \in \mathbb{N}_0, k \geq 3\big\} \cup \big\{y + 2k + 2 \cdot [0, k] : y, k \in \mathbb{N}_0\big\}$.

(4) $\Delta\big(\mathcal{B}_{\pm}(C_2 \oplus C_4)\big) = [1, 2]$, and $\mathcal{L}\big(\mathcal{B}_{\pm}(C_2 \oplus C_4)\big) = \big\{y + k + [0, k] : y \in \mathbb{N}_0, k \geq 2\big\} \cup \big\{y + 2k + 2 \cdot [0, k] : y, k \in \mathbb{N}_0\big\}$.

The characterization problem III

Theorem (Fabsits, Geroldinger, R., Zhong, 2024)

Let G be a finite abelian group, let $n \geq 5$ be an odd integer and let C_n be a cyclic group of order n.
If $\mathcal{L}\big(\mathcal{B}_{\pm}(G)\big) = \mathcal{L}\big(\mathcal{B}_{\pm}(C_n)\big)$, then G and C_n are isomorphic.

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