Integer-Valued Polynomials on Semidomains

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AMS-UMI International Joint Meeting 2024
Università degli Studi di Palermo, Palermo, Italy

July 24, 2024

Structure

1. Background: monoids and semidomains
2. Construction
3. Factorization properties

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An element $a \in M \setminus M^\times$ is called an atom if $a = bc$ for some $b, c \in M$ implies that either $b \in M^\times$ or $c \in M^\times$. We denote by $\mathcal{A}(M)$ the set of atoms of M. A monoid M is atomic provided that every $b \in M \setminus M^\times$ can be expressed as a finite product of atoms.

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Let M be an atomic monoid, and let $b \in M \setminus M^{\times}$.

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$$\text{UFM} \implies \text{FFM} \implies \text{BFM} \implies \text{ACCP} \implies \text{atomic}$$

Background: Semidomains

Definition. A commutative semiring S is a (nonempty) set endowed with two binary operations denoted by '$+$' and '$\cdot$' and called addition and multiplication, respectively, such that the following conditions hold:

1. $(S, +)$ is a commutative monoid with its identity element denoted by 0;
2. $(S, \cdot)$ is a commutative semigroup with an identity element denoted by 1;
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Definition. A semidomain is a subsemiring of an integral domain.

Examples: integral domains, Puiseux semirings, $\mathbb{N}_0$, $\mathbb{R}_0$, $\mathbb{N}_0[X]$, $\mathbb{R}_0[X]$

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For a commutative semiring S, the following conditions are equivalent.

(a) S is a semidomain.

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We say that a semidomain S is atomic (resp., satisfies the ACCP) if its multiplicative monoid S^* is atomic (resp., satisfies the ACCP). In addition, we say that S is a UFS, FFS, or BFS provided that S^* is a UFM, FFM, or BFM, respectively.

Construction

Definition. Let S be a semidomain. We set $\text{Int}(S) := \{f \in \mathcal{F}(S)[X] \mid f(S) \subseteq S\}$ and call the elements of $\text{Int}(S)$ integer-valued polynomials on S. If $S = \mathbb{N}_0$, then we refer to the elements of $\text{Int}(S)$ as natural-valued polynomials.

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A couple of examples of integer-valued polynomials on semidomains:

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2. $\mathcal{G}(\text{Int}(\mathbb{N}_0)) = \text{Int}(\mathbb{Z})$.

   Proof: Let $f \in \text{Int}(\mathbb{Z})$. We can write $f = \sum_{i=0}^{n} c_i \binom{X}{n_i}$, where $c_0, \ldots, c_n \in \mathbb{Z}$. Thus,

   $$f = \sum_{i=0}^{m} c_{t_i} \binom{X}{n_{t_i}} - \sum_{j=0}^{\ell} c_{r_j} \binom{X}{n_{r_j}},$$

   where c_{t_i} and c_{r_j} are positive integers for every $i \in [\![0, m]\!]$ and every $j \in [\![0, \ell]\!]$.

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Proposition (Chapman-Kaplan-P., 202?)

Let $d \in \mathbb{N}$, and let $f \in \text{Int}(\mathbb{N}_0)$ be a random polynomial of degree at most d. The probability that f is in $\mathbb{N}_0[X]$ is $\frac{1}{d!(d-1)!\cdots 2!}$.

Factorization Properties

Theorem (Chapman-Kaplan-P., 202?)

Let S be a semidomain. The following statements hold.

1. $\text{Int}(S)$ satisfies the ACCP if and only if S satisfies the ACCP.
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Is $\text{Int}(S)$ a UFS when S is a UFS?

Factorization Properties

Given an atomic monoid M, the elasticity of a nonunit $b \in M$, denoted by $\rho(b)$, is defined as $\rho(b) = \frac{\sup \mathsf{L}(b)}{\inf \mathsf{L}(b)}$. By convention, we set $\rho(u) = 1$ for every $u \in M^\times$.

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Proposition(Chapman-Kaplan-P., 202?)

The elasticity of $\text{Int}(\mathbb{N}_0)$ is infinite.

It is known that $\text{Int}(\mathbb{Z})$ is fully elastic, so we pose the following question.

Open Question

Is $\text{Int}(\mathbb{N}_0)$ fully elastic?

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Grazie mille!