Bounds for syzigies of monomial curves

(joint work with G. Caviglia and A. Sammartano)

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Preliminaries

Given $g_0 < \ldots < g_e$ integers, with $(g_0, \ldots, g_e) = 1$, a *relation* between them is (informally) an equation of the form

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Problem

How many *minimal* relations are there among $g_0, \ldots, g_e$?

Preliminaries

Let $\Gamma = \langle g_0, \dots, g_e \rangle_{\mathbb{N}} \subseteq \mathbb{N}$.

- $g_0, \dots, g_e$ are the minimal generators of Γ.
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Define $\varphi : \mathbb{N}^{e+1} \to \mathbb{N}$ as $\varphi(\lambda_0, \ldots, \lambda_e) = \lambda_0 g_0 + \ldots + \lambda_e g_e$. Then

$\Gamma \cong \mathbb{N}^{e+1} / \ker \varphi$, where $\ker \varphi = \{(a, b) \in \mathbb{N}^{e+1} \times \mathbb{N}^{e+1} : \varphi(a) = \varphi(b)\}$.

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Example

$\Gamma = \langle 7, 9, 12, 15 \rangle$. Then
$\ker \varphi = \langle [(3, 0, 0, 0), (0, 1, 1, 0)], [(0, 0, 2, 0), (0, 1, 0, 1)],$
$[(0, 3, 0, 0), (0, 0, 1, 1)], [(0, 2, 1, 0), (0, 0, 0, 2)] \rangle$.

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Problem

Compute the number of minimal relations $\rho(\Gamma)$.

Small cases

Notation:

- $edim(\Gamma) = e + 1$ (*embedding dimension*);
- $mult(\Gamma) = g_0$ (*multiplicity*).

We always have $e + 1 \leq g_0$.

- $e = 0 \Rightarrow \Gamma = \mathbb{N}, \rho(\Gamma) = 0$;
- $e = 1 \Rightarrow \Gamma = \langle g_0, g_1 \rangle, \ker \varphi = \langle [(g_1, 0), (0, g_0)] \rangle, \rho(\Gamma) = 1$;
- $e = 2 \Rightarrow \rho(\Gamma) \in \{2, 3\}$ [Herzog, 1970]
- If $e \geq 3$, $\rho(\Gamma)$ can be arbitrarily large; [Bresinsky, 1975]
- $\rho(\Gamma) \geq e$, and if $\rho(\Gamma) = e$ then Γ is called a *complete intersection*.
- $\rho(\Gamma) \leq \binom{g_0}{2} - 2g_0 - 2e + 2 \leq \binom{g_0}{2}$ [Rosales, 1996]

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- $B(e,e+2)=B(e,e+3)=\binom{e+1}{2}$ [García-Sánchez,Rosales, 1998].

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- $C(w) < \infty$. [Vu, 2014]
- Conjecture: $C(w) = \binom{w+1}{2}$. [Herzog,Stamate, 2014]
- No explicit upper bound.

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Semigroup rings

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Example

$\Gamma = \langle 7, 9, 12, 15 \rangle$.
$R_\Gamma = K[[t^7, t^9, t^{12}, t^{15}]] \cong K[[w, x, y, z]]/I_\Gamma$, where

$$I_\Gamma = (w^3 - xy, y^2 - xz, x^3 - yz, x^2y - z^2) \subseteq K[[w, x, y, z]].$$

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- If $Q = K[x_1, \ldots, x_e] \cong gr(\overline{P})$, then $gr(\overline{R}_\Gamma) = Q/\overline{I}_\Gamma^*$, where $\overline{I}_\Gamma^*$ is the ideal of initial forms of $\overline{I}_\Gamma$.

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- If $Q = K[x_1,\dots,x_e] \cong gr(\overline{P})$, then $gr(\overline{R}_\Gamma) = Q/\overline{I}_\Gamma^*$, where $\overline{I}_\Gamma^*$ is the ideal of initial forms of $\overline{I}_\Gamma$.
- If $J_\Gamma = in_{revlex}(\overline{I}_\Gamma^*) \subseteq Q$, then

$$\rho(\Gamma) = \mu(I_\Gamma) = \mu(\overline{I}_\Gamma) \leq \mu(\overline{I}_\Gamma^*) \leq \mu(J_\Gamma).$$

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Since $x^4 = (x(x^3 - yz) - z(xy))^*$ we have
$\overline{I}_\Gamma^* = (xy, y^2 - xz, yz, z^2, x^4) \subseteq K[x, y, z]$

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And similarly, we can see that $x^2z \in J_\Gamma$, thus
$J_\Gamma = (xy, y^2, yz, z^2, x^4, x^2z) \subseteq K[x, y, z]$

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Theorem

$J_\Gamma \subseteq (x_1, \ldots, x_{e-1})^2 + x_e^q(x_1, \ldots, x_e)$, where $q = \left\lfloor \frac{g_0 - 1}{g_e - g_0} \right\rfloor$.
Moreover, $HS(Q/J_\Gamma, d) \leq 1 + d(g_e - g_0)$ for all $d \in \mathbb{N}$.

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Problem 2 (extended)

Let $S = K[x_1, \ldots, x_e]$, and $I \subseteq S$ be a homogeneous ideal such that $\ell(S/I) < \infty$, and $w \in \mathbb{N}$.
Find upper bounds for $\mu(I)$ under the assumption that $HS(S/I, d) \leq 1 + dw$ for all $d \in \mathbb{N}$.

The key idea - Problem 2

A *lexsegment ideal* $L \subseteq S$ is a monomial ideal such that, for each d, the graded component $[L]_d$ is spanned by the first $\dim_K [L]_d$ monomials of $[S]_d$, with respect to the lexicographic order.
For any homogeneous ideal $I \subseteq S$, there exists a unique lexsegment ideal, denoted by $Lex(I) \subseteq S$, such that $HF(I) = HF(Lex(I))$.

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Theorem [Bigatti-Hulett-Pardue]

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If $L \subseteq S$ is a lex ideal, consider $\hat{L} = \frac{L+(x_e)}{x_e} \subseteq \hat{S} = K[x_1, \ldots, x_{e-1}]$.
Then $\mu(L)$ is related to $\ell(\hat{S}/\hat{L})$. By studying this length we obtain:

Theorem

$\mu(I) \leq w \cdot 9^{\sqrt{2w}}$, and thus $C(w) \leq w \cdot 9^{\sqrt{2w}}$.

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Let $L = L(e, m) \subseteq K[x_1, \dots, x_e] = P$ be the unique ideal such that P/L is artinian of length m, $L = (x_1, \dots, x_e)^{r+1} + \tilde{L}$ and $\tilde{L}$ is generated by the first s monomials of degree r in the lex ordering.

Example

$L(3, 16) = (x, y, z)^4 + (x^3, x^2y, x^2z, xy^2) \subseteq K[x, y, z]$.

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Let $\ell(e, m) = \mu(L(e, m))$.

Theorem [Elias, Robbiano, Valla, 1991]

If $\ell(P/I) = m$ then $\mu(I) \leq \ell(e, m)$.

An explicit formula for $\ell(e, m)$

Let n, d be positive integers.
There exist unique integers $n_d > n_{d_1} > \ldots > n_j \geq j \geq 1$ such that

$$n = \binom{n_d}{d} + \ldots + \binom{n_j}{j}.$$

Define

$$n^{<d>} = \binom{n_d + 1}{d + 1} + \ldots + \binom{n_j + 1}{j + 1}.$$

Theorem [Elias, Robbiano, Valla, 1991]

Let $e, m \in \mathbb{N}$ be such that $4 \leq e + 1 \leq m$, let r be the unique integer such that $\binom{e+r-1}{r-1} \leq m < \binom{e+r}{r}$, and let $s = m - \binom{e+r-1}{r-1}$. Then $\ell(e, m) = \binom{e+r-1}{r} + s^{<r>} - s$.

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The answer is yes...

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2. For every $\delta = m - e$ fixed, $B(e, m) = \ell(e, m)$ for $m >> 0$.

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...and no (depending on e and m).

Theorem

1. $B(e, m) < \ell(e, m)$ if $m - e = 7$ and $e = 3, 4, 5$.
2. For every fixed e, $B(e, m) < \ell(e, m)$ for $m >> 0$.

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$J_\Gamma = (xy) + (x, y, z)^3 \subseteq K[x, y, z]$.
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Example

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Some examples

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$\Gamma = \langle 9, 10, 12, 13 \rangle$, $e = 3, m = g_0 = 9$.
$I_\Gamma = (z^3 - wx^3, yz^2 - w^2x^2, y^2z - w^3x, y^3 - w^4, xz^2 - w^4, xy - wz, x^2z - wy^2, x^3 - w^2y) \subseteq K[[w, x, y, z]]$.
$\overline{I}_\Gamma = (z^3, yz^2, y^2z, y^3, xz^2, xy, x^2z, x^3) \subseteq K[[x, y, z]]$.
$J_\Gamma = (xy) + (x, y, z)^3 \subseteq K[x, y, z]$.
On the other hand $L(3, 9) = (x^2) + (x, y, z)^3$.
$\mu(I_\Gamma) = \mu(J_\Gamma) = \mu(L(e, m))$.

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$\Gamma = \langle 10, 11, 13, 14 \rangle, e = 3, m = g_0 = 10$.
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$J_\Gamma = (xy) + (x, y, z)^3 \subseteq K[x, y, z]$.
But $L(3, 10) = (x, y, z)^3$,and $\mu(J_\Gamma) < \mu(L(e, m))$.

Thank you for your attention!