Integer-valued Polynomials Over Subrings of Matrix Algebras

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Notation

- D: commutative integral domain with quotient field K.
- $M_n(D)$: ring of $n \times n$ matrices with entries in D. (Similar for $M_n(K)$).

"Classical" ring of IVP: $\text{Int}_K(D) = \{f \in K[x] \mid f(a) \in D \text{ for all } a \in D\}$ - well understood.

What about plugging in elements in some D-algebra? In particular matrices over D?

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Two flavours of matrix integer-valued polynomials

1. Scalar coefficient polynomials:

$$\mathrm{Int}_K(\mathrm{M}_n(D)) = \{f \in K[x] \mid f(A) \in \mathrm{M}_n(D) \text{ for all } A \in M_n(D)\}.$$

   Form a subring of $K[x]$. Studied by Frisch, Peruginelli, Werner, ...

2. Matrix coefficient polynomials:

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   Where for $f = \sum_i c_i x^i$, we define $f(A) = \sum_i c_i A^i$ (right evaluation). First studied by Werner (2012). Natural question: Do they form a ring? What is their structure?

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Structure of $\text{Int}_{\mathrm{M}_n(K)}(\mathrm{M}_n(D))$

Caution

Right evaluation is not a ring homomorphism in the noncommutative setting!

For $f, g \in \mathrm{M}_n(K)[x]$ and $A \in \mathrm{M}_n(D)$, it could happen that

$$(fg)(A) \neq f(A)g(A),$$

but we still have

$$(f+g)(A) = f(A) + g(A).$$

So $\text{Int}_{\mathrm{M}_n(K)}(\mathrm{M}_n(D))$ is closed under addition, but what about multiplication? I.e. is it a subring of $\mathrm{M}_n(K)[x]$?

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Theorem (Werner, 2012)

$\mathrm{Int}_{\mathrm{M}_n(K)}(\mathrm{M}_n(D))$ *is a subring of* $\mathrm{M}_n(K)[x]$.

For $f, g \in \mathrm{Int}_{\mathrm{M}_n(K)}(\mathrm{M}_n(D))$, need to show that fg is also integer-valued. While $(fg)(A) \neq f(A)g(A)$, we still have $(fg)(A) = (fg(A))(A)$, so only need to consider the products fC for $C \in \mathrm{M}_n(D)$.

If C is a unit, everything works out nicely; same for sums of units. As every matrix ($n \geq 2$) is a sum of units, we are done.

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Structure of $\mathrm{Int}_{\mathrm{M}_n(K)}(\mathrm{M}_n(D))$

Theorem (Frisch, 2013)

$$\mathrm{Int}_{\mathrm{M}_n(K)}(\mathrm{M}_n(D)) \cong \begin{bmatrix} \mathrm{Int}_K(\mathrm{M}_n(D)) & \mathrm{Int}_K(\mathrm{M}_n(D)) & \cdots & \mathrm{Int}_K(\mathrm{M}_n(D)) \\ \mathrm{Int}_K(\mathrm{M}_n(D)) & \mathrm{Int}_K(\mathrm{M}_n(D)) & \cdots & \mathrm{Int}_K(\mathrm{M}_n(D)) \\ \vdots & \vdots & \ldots & \vdots \\ \mathrm{Int}_K(\mathrm{M}_n(D)) & \mathrm{Int}_K(\mathrm{M}_n(D)) & \cdots & \mathrm{Int}_K(\mathrm{M}_n(D)). \end{bmatrix}$$

Upper triangular matrices

Theorem (Frisch, 2017)

Let $\mathsf{T}_n(D)$ *denote the ring of* $n \times n$ *upper triangular matrices. Then* $\mathrm{Int}_{\mathsf{T}_n(K)}(\mathsf{T}_n(D)) = \{f \in \mathsf{T}_n(K)[x] \mid f(A) \in \mathsf{T}_n(D) \text{ for all } A \in \mathsf{T}_n(D)\}$ *is a subring of* $\mathsf{T}_n(K)[x]$ *and is isomorphic to*

$$\begin{bmatrix} \mathrm{Int}_K(\mathsf{T}_n(D)) & \mathrm{Int}_K(\mathsf{T}_{n-1}(D)) & \cdots & \mathrm{Int}_K(\mathsf{T}_2(D)) & \mathrm{Int}_K(D) \\ 0 & \mathrm{Int}_K(\mathsf{T}_{n-1}(D)) & \cdots & \mathrm{Int}_K(\mathsf{T}_2(D)) & \mathrm{Int}_K(D) \\ & & \ddots & & \\ 0 & 0 & \cdots & \mathrm{Int}_K(\mathsf{T}_2(D)) & \mathrm{Int}_K(D) \\ 0 & 0 & \cdots & 0 & \mathrm{Int}_K(D) \end{bmatrix}.$$

Block triangular matrices

Theorem (Sedighi Hafshejani, Naghipour, Rismanchian, 2019)

Let $0 \leq s \leq n$ *and denote by* ${}_{\mathsf{s}}\mathsf{L}_{\mathsf{n}}(D)$ *the ring of square matrices with entries in* D *which are of the form*

$$\begin{bmatrix} \overbrace{\begin{matrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{matrix}}^{s} & \overbrace{\begin{matrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \end{matrix}}^{n-s} \end{bmatrix}.$$

Then $\mathrm{Int}_{{}_{\mathsf{s}}\mathsf{L}_{\mathsf{n}}(K)}({}_{\mathsf{s}}\mathsf{L}_{\mathsf{n}}(D))$ *is a subring of* ${}_{\mathsf{s}}\mathsf{L}_{\mathsf{n}}(K)[x]$.

Question

For which other subrings of $\mathrm{M}_n(D)$ can we do similar things and obtain *rings* of integer-valued polynomials?

Definition of $\mathrm{M}_{\precsim}(D)$

$\mathrm{M}_n(D)$, $\mathrm{T}_n(D)$ and ${}_{\mathrm{s}}\mathrm{L}_{\mathrm{n}}(D)$ are all instances of the following construction: Fix certain positions (i, j) and take all matrices whose only nonzero entries are in these positions. What possible patterns for these positions are allowed?

Definition

Let $\precsim$ be a preorder on $\{1, \ldots, n\}$, i.e. a reflexive and transitive relation. Define a subring $\mathrm{M}_{\precsim}(D)$ of $\mathrm{M}_n(D)$ by setting

$$\mathrm{M}_{\precsim}(D) = \{(a_{ij}) \in \mathrm{M}_n(D) \mid a_{ij} \neq 0 \Rightarrow i \precsim j\}.$$

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Example

$$\begin{bmatrix} * & * & 0 & * & * & * & * & * & 0 & * \\ 0 & * & 0 & * & 0 & 0 & * & 0 & 0 & * \\ 0 & * & * & * & 0 & 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * \\ * & * & 0 & * & * & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & 0 & 0 & * & 0 & 0 & * \\ * & * & 0 & * & * & * & * & * & 0 & * \\ 0 & * & * & * & 0 & 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} \cong \begin{bmatrix} * & * & * & 0 & 0 & * & * & * & * & * \\ * & * & * & 0 & 0 & * & * & * & * & * \\ * & * & * & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

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A bigger example

$$
\begin{bmatrix}
* & * & * & 0 & 0 & * & * & * & * & 0 & 0 & * & * & * & * \\
* & * & * & 0 & 0 & * & * & * & * & 0 & 0 & * & * & * & * \\
* & * & * & 0 & 0 & * & * & * & * & 0 & 0 & * & * & * & * \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & *
\end{bmatrix}
$$

A useful property of $\mathrm{M}_{\precsim}(D)$

Lemma

Let $\precsim$ be a preorder on $\{1, \ldots, n\}$ and $A \in \mathrm{M}_{\precsim}(D)$. Then A can be written as $A = B + C$ where B is a sum of units and C is a diagonal matrix whose only nonzero entries are in positions (h, h) where the equivalence class $[h]_{\sim}$ is a singleton.

Proof.

We get everything off the diagonal from elementary matrices and square matrices of size ≥ 2 are sum of units. The only thing remaining is exactly the matrix C claimed above. $\square$

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Theorem

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$\mathrm{Int}_{\mathrm{M}_{\precsim}(K)}(\mathrm{M}_{\precsim}(D)) = \{f \in \mathrm{M}_{\precsim}(K)[x] \mid f(A) \in \mathrm{M}_{\precsim}(D)$ *for all* $A \in \mathrm{M}_{\precsim}(D)\}$*.*
Then $\mathrm{Int}_{\mathrm{M}_{\precsim}(K)}(\mathrm{M}_{\precsim}(D))$ *is a subring of* $\mathrm{M}_{\precsim}(K)[x]$*.*

Main proof ingredients:

1. Suffices to show that $\mathrm{Int}_{\mathrm{M}_{\precsim}(K)}(\mathrm{M}_{\precsim}(D))$ is closed under multiplication by elements of $\mathrm{M}_{\precsim}(D)$ from the right.
2. $\mathrm{M}_{\precsim}(K)[x] \cong \mathrm{M}_{\precsim}(K[x])$; give characterization of $f = (f_{ih}) \in \mathrm{Int}_{\mathrm{M}_{\precsim}(K)}(\mathrm{M}_{\precsim}(D))$ in terms of the scalar polynomials f_{ih}. (which is particularly nice if $[h]_{\sim}$) is singleton).
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The structure of $\mathrm{Int}_{\mathrm{M}_{\precsim}(K)}(\mathrm{M}_{\precsim}(D))$

Knowing that $\mathrm{Int}_{\mathrm{M}_{\precsim}(K)}(\mathrm{M}_{\precsim}(D))$ is a ring, one can also obtain a nicer characterization of its elements in terms of scalar-coefficient integer-valued polynomials.

Connection to null-polynomials

Let $f \in \mathsf{M}_{\precsim}(K)[x]$ and write $f = g/d$ for $g \in \mathsf{M}_{\precsim}(D)[x]$ and $d \in D$. Then

$$f \in \mathrm{Int}_{\mathsf{M}_{\precsim}(K)}(\mathsf{M}_{\precsim}(D)) \Leftrightarrow \bar{g}(A) = 0 \in \mathsf{M}_{\precsim}(D/dD) \text{ for all } A \in \mathsf{M}_{\precsim}(D/dD)$$

For any finite (possibly noncommutative) ring R, there are nonzero polynomials vanishing everywhere (null-polynomials). Consider the set of them: $\mathsf{N}(R) = \{f \in R[x] \mid f(r) = 0 \text{ for all } r \in R\}$. For which R is this a two-sided ideal of $R[x]$?

True for many classes: semisimple, odd order, ... (Werner, 2013).

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Conjecture (Werner, 2013)

For any finite (noncommutative) ring R, the set of null-polynomials $\mathrm{N}(R)$ is a two sided ideal.

It suffices to show the theorem for all subrings of matrix rings over certain commutative finite rings.

Theorem

Let $\precsim$ be any preorder and S any commutative ring. Then the set $\mathrm{N}(\mathrm{M}_{\precsim}(S))$ is a two-sided ideal of $\mathrm{M}_{\precsim}(S)[x]$.

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Conclusion

- For any preorder $\precsim$ on $\{1, \ldots, n\}$, define rings $\mathrm{M}_{\precsim}(D) \subset \mathrm{M}_n(D)$ by restricting entries in certain positions to be 0.
- Integer-valued polynomials $\mathrm{Int}_{\mathrm{M}_{\precsim}(K)}(\mathrm{M}_{\precsim}(D))$ form a subring of $\mathrm{M}_{\precsim}(K)[x]$.
- There is a characterization of $\mathrm{Int}_{\mathrm{M}_{\precsim}(K)}(\mathrm{M}_{\precsim}(D))$ in terms of polynomials with coefficients in K.
- The set of null-polynomials $\mathrm{N}(\mathrm{M}_{\precsim}(S))$ is an ideal for all commutative rings S.
- What about restricting the entries to be in some ideal of D instead of zero?

Thank you for your attention!

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- There is a characterization of $\mathrm{Int}_{\mathrm{M}_{\precsim}(K)}(\mathrm{M}_{\precsim}(D))$ in terms of polynomials with coefficients in K.
- The set of null-polynomials $\mathrm{N}(\mathrm{M}_{\precsim}(S))$ is an ideal for all commutative rings S.
- What about restricting the entries to be in some ideal of D instead of zero?

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