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Egyptian integral domains and the ring of reciprocals

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1. Egyptian integral domains.

An Egyptian fraction

$$q = \frac{1}{a_1} + \ldots + \frac{1}{a_n}$$

is a representation of a rational number q as a sum of distinct unit fractions.

It is well-known since ancient times that any rational number q can be represented as an Egyptian fraction (even with arbitrarily large denominators).

Definition (LG - Loper - Oman). Let D be an integral domain. An element $x \in D$ is *Egyptian* if there exist distinct nonzero elements $d_1, \ldots, d_n \in D$ such that

$$x = \frac{1}{d_1} + \ldots + \frac{1}{d_n}.$$

The ring D is an *Egyptian domain* if all its nonzero elements are Egyptian.

Remark. Thanks to the fact that integers have Egyptian representations with arbitrarily large denominators, the assumption of distinct denominators is redundant for integral domains.

Epstein extended this definition also to rings with zerodivisors.

The ring of integers $\mathbb{Z}$ is an Egyptian domain. Any overring of an Egyptian domain is still an Egyptian domain.

The polynomial ring $K[X]$ is not an Egyptian domain (X is not an Egyptian element).

It is easy to observe that:

- Units of D are Egyptian elements.
- If $x, y \in D$ are Egyptian, then $x + y$, xy are Egyptian.
- If $x, y \in D$ and xy is Egyptian, then x, y are Egyptian.

Using this it follows that integral domains with nonzero Jacobson radical are Egyptian. For this pick nonzero elements $x \in D$, $y \in J(D) \setminus \{0\}$ and use that $xy = (xy + 1) - 1$ is a sum of units, hence Egyptian.

Some other results:

- Every proper overring of $K[X]$ is Egyptian.
- Integral extensions of Egyptian domains are Egyptian.
- Certain ultraproducts of infinite copies of $\mathbb{Z}$ are not Egyptian.
- A semigroup algebra $K[S]$ is Egyptian only if S is a group (Epstein).

The question of characterizing Egyptian domains is still open. A good tool for this study is the ring of reciprocals.

2. The ring of reciprocals

We know that $\mathbb{Z}$ and $K[X]$ are both Euclidean domains, but one is Egyptian and the other is not. Motivated by understanding this difference, Epstein gave this definition:

Definition (Epstein). Let D be an integral domain with quotient field $\mathcal{Q}(D)$. The *ring of reciprocals* $R(D)$ is the subring of $\mathcal{Q}(D)$ generated by all fractions $\frac{1}{d}$ for nonzero $d \in D$.

Remark. D is Egyptian if and only if $R(D) = \mathcal{Q}(D)$.

We have

$$R(\mathbb{Z}) = \mathbb{Q}, \quad R(K[X]) = K[X^{-1}]_{(X^{-1})}.$$

Theorem (Epstein).
If D is an Euclidean domain, then $R(D)$ is either a field or a DVR.

It is interesting now to study $R(D)$ for arbitrary (non-Egyptian) D.

Let D be an integral domain with quotient field $\mathcal{Q}(D)$.

Let E be the subset of the Egyptian elements of D (it is multiplicatively closed). Let K be the quotient field of the ring $E \cup \{0\}$.

Theorem (Epstein, LG, Loper).

- $R(D) = R(E^{-1}D)$.
- $E^{-1}D$ is a K-algebra whose Egyptian elements are only the nonzero elements of K.
- $R(D)$ is always a local domain with maximal ideal generated by $\frac{1}{x}$ for $x \in E^{-1}D \setminus K$.

3. The ring of reciprocals of a polynomial ring in several variables.

In joint work with Epstein and Loper, we studied $R_n := R(K[X_1, \ldots, X_n])$.

This ring has a "polynomial-like behavior":

- For $i < n$, $R_n \cap K(X_1, \ldots, X_i) = R_i$.
- There exists prime ideals $\mathfrak{q}_i$ such that

$$\frac{R_n}{\mathfrak{q}_i} \cong R_i, \quad (R_n)_{\mathfrak{q}_i} = R(K(X_1, \ldots, X_i)[X_{i+1}, \ldots, X_n]).$$

- $\dim(R_n) = n$.
- R_n has infinitely many prime ideals of every height $i = 1, \ldots, n-1$.

But also a very different behavior:

- The element $\frac{1}{X_1 X_2 \cdots X_n} \in \mathfrak{p}$ for every nonzero prime $\mathfrak{p}$ of R_n.
- For $n \geq 2$, R_n is not Noetherian.
- For $n \geq 2$, R_n is not integrally closed.
- If $n = 2$, every finitely generated ideal of R_2 is contained in all but finitely many primes but all the localizations at non-maximal primes are Noetherian.

Open question. Describe the elements of the integral closure of R_n, establish whether is local, completely integrally closed, etc..

4. Factorization.

Theorem. The ring R_n is atomic.

Idea of the proof. Let $R_n^* = \sigma(R_n)$, where $\sigma : K(X_1, \ldots, X_n) \to K(X_1, \ldots, X_n)$ is the automorphism mapping $X_i \to \frac{1}{X_i}$.

Then, R_n^* is isomorphic to R_n and is an overring of $K[X_1, \ldots, X_n]$.

Let V be the order valuation ring defined by the powers of the maximal ideal $(X_1, \ldots, X_n)$ of $K[X_1, \ldots, X_n]$.

One can check that R_n^* is dominated by V. It follows that it must be atomic.

Factorization in R_n is not unique:

$$\frac{1}{X} \cdot \frac{1}{Y} = \frac{1}{X+Y} \cdot \left(\frac{1}{X} + \frac{1}{Y}\right).$$

5. Semigroup algebras.

Open question. For an integral domain D, is $\dim(R(D)) \leq \dim D$?

The answer is yes if $D = K[S]$ for S a submonoid of the positive part $G_{\geq 0}$ of a totally ordered abelian group G.

In this setting is also possible to construct for every $n \geq m \geq 0$ an integral domain D such that $\dim(D) = n$ and $\dim(R(D)) = m$.

Example (LG).

$D = K[\{YX^k\}_{k \in \mathbb{Z}}]$ has Krull dimension 2.

$R(D) \cong K + YK(X)[Y]_{(Y)}$ has Krull dimension 1.

A similar construction involving more variables yields examples where $\dim(D)$ and the difference $\dim(D) - \dim(R(D))$ are arbitrarily large.

Let $S \subsetneq \mathbb{N}$ be a numerical semigroup. Let S' be the semigroup generated by all differences $ns - s_1 - s_2 - \ldots - s_{n-1}$ for $n \geq 1$, $s, s_1, \ldots, s_{n-1} \in S$ and $s_i < s$.

We have $S \subseteq S' = \langle g_1, \ldots, g_t \rangle \subsetneq \mathbb{N}$.

Theorem (LG).

$$R(K[S]) = K[X^{-g_1}, \ldots, X^{-g_t}]_{(X^{-g_1}, \ldots, X^{-g_t})}.$$

A similar result holds more in general for any semigroup S having $G_{\geq 0}$ as integral closure.

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Thanks for your attention!