On an Inverse Zero-sum Question for Rank Two Groups

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The Davenport Constant

Definition (Davenport Constant)

For a finite abelian group G, the **Davenport Constant** $\mathrm{D}(G)$ is the minimal integer such that any sequence of $\mathrm{D}(G)$ terms from G must have a nontrivial **zero-sum** subsequence (one whose terms sum to zero).

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Remark (Relation to Factorization Theory)

Many arithmetic questions of in a monoid M, for example any (Transfer) Krull Monoid, including Krull Domains, can be directly translated into questions about zero-sum sequences over finite abelian groups.

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$$e_1^{n-1} \cdot e_2^{n-1}$$

shows $\mathsf{D}(G) \geq 2n - 1$.

- **Notation:** Sequences written as unordered multiplicative strings, so

  $$S = g_1 \cdot \ldots \cdot g_\ell$$

  with $g_1, \ldots, g_\ell \in G$ is a sequence of terms from G with length ℓ, and

  $$g^n = \underbrace{g \cdot \ldots \cdot g}_{n}.$$

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- Group Algebra/Polynomial Methods used to show $\eta(C_p^2) = 3p - 2$
- Inductive argument using $\mathrm{D}(G)$ and $\eta(G)$ completes the proof and also shows $\eta(C_n \oplus C_n) = 3n - 2$.

Characterization of Extremal Sequences I

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Characterize all sequences of length $\mathsf{D}(C_n^2) - 1 = 2n - 2$ *that do NOT have a nontrivial zero-sum sequence.*

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Equivalent to characterizing all **minimal** *zero-sum sequences of length* $\mathsf{D}(C_n^2) = 2n - 1$*, so those zero-sum sequences having no proper, nontrivial zero-sum subsequence.*

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- Question Resolved: All have the form

$$S = e_1^{n-1} \prod_{i=1}^{n} (x_i e_1 + e_2)$$

for some basis (e_1, e_2) for $G = C_n^2$ and some $x_1, \ldots, x_n \in [0, n-1]$ with $x_1 + \ldots + x_n \equiv 1 \mod n$.

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- Total Length nearly 150 pages.

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$\mathsf{s}_{\leq 2n-1-k}(G) = 2n - 1 + k$ *for* $G = C_n^2$ *and* $k \in [0, n-1]$.

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Remark

$\mathsf{s}_{\leq 2n-1-k}(G) = \infty$ *for* $k \geq n$,
$\mathsf{s}_{\leq 2n-1-k}(G) = \eta(G)$ *for* $k = n-1$,
$\mathsf{s}_{\leq 2n-1-k}(G) = \mathsf{D}(G)$ *for* $k \leq 0$,

Characterization of Extremal Sequences III

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Characterize all sequences of length $\mathsf{s}_{\leq 2n-1-k}(G) - 1 = 2n - 2 + k$ *that do NOT have a nontrivial zero-sum sequence of length at most* $2n - 1 - k$.

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Conjecture

When $k \in [2, n-2]$, *all such sequences have the form*

$$S = e_1^{n-1} \cdot e_2^{n-1} \cdot (e_1 + e_2)^k$$

for some basis (e_1, e_2) *for* $G = C_n^2$.

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Characterization of Extremal Sequences IV

- (G., Wang, Zhao 2020) Shown true when $n = p$ is prime and $k \leq \frac{2p-1}{3}$.
- (G.,Liu 2022) Reduction to case $n = p$ prime
- Final piece $n = p$ now resolved:

Theorem (Ebert, G. 2023)

Let $G = C_p \oplus C_p$ with p a prime, let $k \in [2, p-2]$ be an integer, and let S be a sequence of terms from G with $|S| = \mathsf{D}(G) + k - 1 = 2p - 2 + k$ having no nonempty zero-sum subsequence of length at most $\mathsf{D}(G) - k = 2p - 1 - k$. Then there is a basis (e_1, e_2) for G such that

$$S = e_1^{p-1} \cdot e_2^{p-1} \cdot (e_1 + e_2)^k.$$

Ideas for the Proof I

- First Step: Extend a Lemma from [G., Wang, Zhao] for $k \leq \frac{2p-1}{3}$ to general $k \leq p-1$.

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Lemma

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$$\text{Number of zero-sums of length } 2p-1 \equiv k \mod p.$$

In particular, S contains at least k zero-sum subsequences of length $\mathsf{D}(G) = 2p-1$, and any such zero-sum is minimal.

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- Using the above lemma and the characterization for $\mathsf{D}(G)$ gives the complete structure for $2p-1$ terms in S. In particular, there is one term e_1 with multiplicity $p-1$.
- Matrix calculation modulo p using binomial congruences.

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- Take the sequence $S = g_1 \cdot \ldots \cdot g_{2p-1+k}$ in $G = C_p^2 = \langle e_1 \rangle \oplus \langle e_2 \rangle$ and create new zero-sum sequence

$$\tilde{S} = (g_1 + e_3) \cdot \ldots \cdot (g_{2p-1+k} + e_3) \cdot (-e_3)^{p-k-1}$$

of terms in $C_p^3 = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_3 \rangle$.

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  of terms in $C_p^3 = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_3 \rangle$.
- Use that $\mathsf{D}(C_p^3) = 3p-2$ is known for primes to find a zero-sum subsequence of $\tilde{S}$, which then corresponds to a zero-sum sequence in S of the target length.

Ideas for the Proof III

- Modify the proof when $S = g_1 \cdot \ldots \cdot g_{2p-2+k}$ has one term less to create a zero-sum sequence of terms

$$\tilde{S} = (g_1 + e_3) \cdot \ldots \cdot (g_{2p-2+k} + e_3) \cdot (-e_3)^{p-k-1} \cdot g_0$$

from C_p^3 of length $\mathsf{D}(C_p^3) = 3p - 2$.

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- Get a 2nd term with multiplicity $p - 1$:

$$S = e_1^{p-1} \cdot e_2^{p-1} \cdot T$$

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- Use another reduction map to a cyclic group to show all terms in T are equal and the proof then concludes.

Thanks!