Asymptotic Behaviour of the v-number of homogeneous ideals

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Two giants in Commutative Algebra

Wolmer Vasconcelos **Jürgen Herzog**

Primary decomposition in Noetherian rings

Let R be a Noetherian ring, and let $I \subset R$ be an ideal. Let

$$I \;=\; Q_1 \cap Q_2 \cap \cdots \cap Q_t$$

be an irredundant primary decomposition of I, where each Q_i is a $\mathfrak{p}_i$-primary ideal.

The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ is uniquely determined for it is equal to the set

$$\{(I : f) \mid f \in R \text{ and } (I : f) \text{ is a prime ideal}\}.$$

We denote this set by $\mathrm{Ass}(I)$.

An example

Let $S = \mathbb{Q}[x, y, z]$ and $I = (xy, x - yz)$. Then

$$I = (y^2, x - yz) \cap (x, z)$$

and $\text{Ass}(I) = \{(x, y), (x, z)\}$.

We have

$$(I : x) = (x, y),$$

and

$$(I : y^2) = (x, z).$$

Graded rings

Let K be a field and let $S = K[x_1, \dots, x_n]$ be the polynomial ring.

We set $\deg x_i = 1$ for $1 \leq i \leq n$. Then, S becomes a graded ring

$$S = \bigoplus_{d \geq 0} S_d$$

where S_d is the K-vector space with basis all monomials of degree d.

A polynomial $f \in S_d$ is called homogeneous of degree d.

An ideal $I \subset S$ is called homogeneous or graded ideal if it can be generated by homogeneous polynomials.

Graded version of the primary decomposition

Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal.

Then each $\mathfrak{p} \in \mathrm{Ass}(I)$ is a graded prime ideal and there exists a homogeneous polynomial $f \in S$ such that $(I : f) = \mathfrak{p}$.

Example. Let $I = (x^3, x^2y, x^2z^2, yz, z^4) \subset S = K[x, y, z]$. Then

$$\mathrm{Ass}(I) = \{(x, z), (x, y, z)\}.$$

In particular, $(I : xy) = (x, z)$ and $(I : x^2z) = (x, y, z)$.

The v-number of a graded ideal

In

S.M. Cooper, A. Seceleanu, S.O. Tohăneanu, M. Vaz Pinto, R.H. Villarreal. *Generalized minimum distance functions and algebraic invariants of geramita ideals*. Adv. Appl. Math., 112:101940, 2020.

the concept of v-number was introduced, in connection with the theory of Reed–Muller type codes and minimum distance functions.

Let $I \subset S$ be a graded ideal and $\mathfrak{p} \in \text{Ass}(I)$.

The $\mathrm{v}_{\mathfrak{p}}(I)$-number of I is defined as

$$\mathrm{v}_{\mathfrak{p}}(I) = \min\{\deg(f) \ : \ f \in S \text{ is homogeneous and } (I : f) = \mathfrak{p}\}.$$

The v-number of I (**v in honor of Vasconcelos**) is defined as

$$\mathrm{v}(I) = \min_{\mathfrak{p} \in \text{Ass}(I)} \mathrm{v}_{\mathfrak{p}}(I).$$

How to compute the v-number of a graded ideal

Let $\mathfrak{m} = (x_1, \dots, x_n)$ be the graded maximal ideal of S.

For a graded module $M = \bigoplus_{d \geq 0} M_d$ we set $\alpha(M) = \min\{d : M_d \neq 0\}$ and $\omega(M) = \max\{d : (M/\mathfrak{m}M)_d \neq 0\}$.

Theorem (Grisalde-Reyes-Villarreal, 2019)

Let $I \subset S$ be a graded ideal and let $\mathfrak{p} \in \mathrm{Ass}(I)$. The following hold.

1. If $\mathcal{G} = \{\overline{g_1}, \dots, \overline{g_r}\}$ is a homogeneous minimal generating set of $(I : \mathfrak{p})/I$, then
$$\mathrm{v}_{\mathfrak{p}}(I) = \min\{\deg(g_i) \ : \ 1 \leq i \leq r \text{ and } (I : g_i) = \mathfrak{p}\}.$$
2. $\mathrm{v}(I) = \min\{\mathrm{v}_{\mathfrak{p}}(I) : \mathfrak{p} \in \mathrm{Ass}(I)\}$.
3. $\mathrm{v}_{\mathfrak{p}}(I) \geq \alpha((I : \mathfrak{p})/I)$, with equality if $\mathfrak{p} \in \mathrm{Max}(I)$.
4. If $\mathrm{Ass}(I) = \mathrm{Max}(I)$, then $\mathrm{v}(I) = \min\{\alpha((I : \mathfrak{p})/I) : \mathfrak{p} \in \mathrm{Ass}(I)\}$.

Powers of ideals and their history

Theorem (Brodmann, 1979)

Let I be an ideal of a Noetherian ring R. Then

$$\text{Ass}(I^{k+1}) = \text{Ass}(I^k)$$

for all $k \gg 0$.

Theorem (Cutkosky-Herzog-Trung, Kodiyalam, 1999)

Let $I \subset S$ be a graded ideal, and let

$$\text{reg}(I) = \max\{i - j : \beta_{i,i+j}(I) \neq 0\}$$

be its Castelnuovo–Mumford regularity.
Then $\text{reg}(I^k) = ak + b$ is a linear function in k for $k \gg 0$.

Consequences of Brodmann result

The common set $\operatorname{Ass}(I^{k+1}) = \operatorname{Ass}(I^k)$ for $k \gg 0$, is called the set of stable primes of I and it is denoted by $\operatorname{Ass}^\infty(I)$.

Consequences

1. The set $\bigcup_{k\geq 1} \operatorname{Ass}(I^k)$ is finite.
2. If $I \subset S$ is a graded ideal, then

$$\mathrm{v}(I^k) = \min_{\mathfrak{p}\in \operatorname{Ass}^\infty(I)} \mathrm{v}_{\mathfrak{p}}(I^k),$$

for all $k \gg 0$.

A blowup algebra associated to stable primes

In

A. Ficarra, E. Sgroi. *Asymptotic Behaviour of the* v*-number of homogeneous ideals*, 2023, preprint arXiv:2306.14243.

we studied the asymptotic behaviour of the function $\text{v}(I^k)$.

Let $I \subset S$ be a graded ideal and let $\mathfrak{p} \in \text{Ass}^\infty(I)$.

Let $\mathcal{F}_\mathfrak{p}(I) = \bigoplus_{d\geq 0}(I^k/\mathfrak{p}I^k)$. Then $\mathcal{F}_\mathfrak{p}(I)$ is a graded ring.

Furthermore, let

$$\text{Soc}_\mathfrak{p}(I) = \bigoplus_{d\geq 0}(I^k : \mathfrak{p})/I^k.$$

Theorem (F-Sgroi 2023)

$\text{Soc}_\mathfrak{p}(I)$ is a finitely generated bigraded $\mathcal{F}_\mathfrak{p}(I)$-module.

Consequences of the finite generation of $\mathrm{Soc}_{\mathfrak{p}}(I)$

Corollary (F-Sgroi 2023)

Let $I \subset S$ be a graded ideal and let $\mathfrak{p} \in \mathrm{Ass}^{\infty}(I)$. Then,

1. $(I^k : \mathfrak{p}) = I(I^{k-1} : \mathfrak{p})$ for all $k \gg 0$.
2. $\mathrm{v}_{\mathfrak{p}}(I^{k-1}) + \alpha(I) \leq \mathrm{v}_{\mathfrak{p}}(I^k) \leq \mathrm{v}_{\mathfrak{p}}(I^{k-1}) + \omega(I)$ for all $k \gg 0$.
3. $\mathrm{v}(I^{k-1}) + \alpha(I) \leq \mathrm{v}(I^k) \leq \mathrm{v}(I^{k-1}) + \omega(I)$ for all $k \gg 0$.
4. $\omega((I^k : \mathfrak{p})/I^k) \geq \alpha((I^{k-1} : \mathfrak{p})/I^{k-1}) + \alpha(I)$, for all $k \gg 0$.
5. $\alpha((I^k : \mathfrak{p})/I^k) \leq \omega((I^{k-1} : \mathfrak{p})/I^{k-1}) + \omega(I)$, for all $k \gg 0$.
6. The function $\mathrm{v}_{\mathfrak{p}}(I^k)$ is strictly increasing, for all $k \gg 0$.
7. The function $\mathrm{v}(I^k)$ is strictly increasing, for all $k \gg 0$.

Asymptotic behaviour of the v-number

Defining a suitable quotient module $\text{Soc}^*_{\mathfrak{p}}(I)$ of $\text{Soc}_{\mathfrak{p}}(I)$, we proved:

Theorem (F-Sgroi 2023)

Let $I \subset S$ be a graded ideal. Then:

1. For all $k \gg 0$ and all $\mathfrak{p} \in \text{Ass}^\infty(I)$, $\text{v}_{\mathfrak{p}}(I^k)$ and $\text{v}(I^k)$ are linear functions in k.
2. For any $\mathfrak{p} \in \text{Ass}^\infty(I)$, $\lim_{k\to\infty} \frac{\text{v}_{\mathfrak{p}}(I^k)}{k}$ exists and belongs to the set $\{d : (I/\mathfrak{m}I)_d \neq 0\}$. In particular $\alpha(I) \leq \lim_{k\to\infty} \frac{\text{v}_{\mathfrak{p}}(I^k)}{k} \leq \omega(I)$.
3. $\lim_{k\to\infty} \frac{\text{v}(I^k)}{k}$ exists and it is equal to $\alpha(I)$.
4. For all $k \gg 0$, $\text{v}(I^k) = \alpha(I)k + b$ for some integer $b \in \mathbb{Z}$.
5. Furthermore, if I is a monomial ideal, then $b \geq -1$.

A conjecture

We expect that

A BIG Conjecture

Let $I \subset S$ be a graded ideal. Then

$$\mathrm{v}(I^k) < \mathrm{reg}(I^k), \quad \text{for all} \quad k \gg 0.$$

The v-number of monomial ideals in two variables

Any monomial ideal $I \subset S = K[x,y]$ is determined by two sequences

$$\mathbf{a} : a_1 > a_2 > \cdots > a_m \geq 0, \qquad \mathbf{b} : 0 \leq b_1 < b_2 < \cdots < b_m$$

such that $G(I) = \{x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \dots, x^{a_m}y^{b_m}\}$. Thus, we set $I = I_{\mathbf{a},\mathbf{b}}$.

Theorem (F-Sgroi 2023)

$$\mathrm{v}(I) = \begin{cases} \min\{a_i + b_{i+1} - 2 \ : \ 1 \leq i \leq m-1\}, & \text{if } b_1 = 0 \text{ and } a_m = 0, \\ \min\left\{\substack{a_1+b_1-1, \\ a_i+b_{i+1}-2} \ : \ 1 \leq i \leq m-1\right\}, & \text{if } b_1 \neq 0 \text{ and } a_m = 0, \\ \min\left\{\substack{a_m+b_m-1, \\ a_i+b_{i+1}-2} \ : \ 1 \leq i \leq m-1\right\}, & \text{if } b_1 = 0 \text{ and } a_m \neq 0, \\ \min\left\{\substack{a_1+b_1-1, \\ a_m+b_m-1, \\ a_i+b_{i+1}-2} \ : \ 1 \leq i \leq m-1\right\}, & \text{otherwise.} \end{cases}$$

The v-number of monomial ideals in two variables

Proposition (F-Sgroi, 2023)

1. $\text{Ass}(I_{\mathbf{a},\mathbf{b}}^k) = \text{Ass}^\infty(I_{\mathbf{a},\mathbf{b}})$ for all $k \geq 1$.
2. $(x) \in \text{Ass}^\infty(I_{\mathbf{a},\mathbf{b}})$ if and only if $a_m > 0$. In this case, for all $k \geq 1$
$$\text{v}_{(x)}(I_{\mathbf{a},\mathbf{b}}^k) = (a_m + b_m)k - 1.$$
3. $(y) \in \text{Ass}^\infty(I_{\mathbf{a},\mathbf{b}})$ if and only if $b_1 > 0$. In this case, for all $k \geq 1$
$$\text{v}_{(y)}(I_{\mathbf{a},\mathbf{b}}^k) = (a_1 + b_1)k - 1.$$
4. $\mathfrak{m} = (x, y) \in \text{Ass}^\infty(I_{\mathbf{a},\mathbf{b}})$ if and only if $m > 0$.

Theorem (F-Sgroi, 2023)

For any integer $a \geq 1$ and $b \geq -1$, there exists a monomial ideal $I_{\mathbf{a},\mathbf{b}}$ such that $\text{v}(I_{\mathbf{a},\mathbf{b}}^k) = ak + b$ for all $k \geq 1$.

The v-number of monomial ideals with linear powers

Let I be a monomial ideal. Then $\text{reg}(I^k) \geq \alpha(I)k$ for all $k \geq 1$.
We say that I has linear powers if $\text{reg}(I^k) = \alpha(I)k$ for all $k \geq 1$.

Proposition (F, 2023)

If I is a monomial ideal, $\text{v}(I^k) \geq \alpha(I)k - 1$ for all $k \geq 1$.

Thus, we expect that

Conjecture (F, 2023)

Let I be a monomial ideal with linear powers. Then, for all $k \geq 1$,

$$\text{v}(I^k) = \alpha(I)k - 1.$$

Some partial results

Theorem (F 2023)

Let $I \subset S$ be a monomial ideal having linear powers. Then

$$\mathsf{v}(I^k) = \alpha(I)k - 1$$

for all $k \geq 1$, in the following cases:

1. $I = I(G)$ is the edge ideal of a cochordal graph. That is, I is a squarefree monomial ideal generated in degree two and having a linear resolution.
2. I is polymatroidal.
3. I is an Hibi ideal.
4. $\operatorname{depth} S/I = 0$.

Hibi ideals

Let $(P, \succeq)$ be a poset, $P = \{p_1, \dots, p_n\}$. A poset ideal $\mathcal{I}$ of P, is a subset of P such that if $p_i \in P$, $p_j \in \mathcal{I}$ and $p_i \preceq p_j$, then $p_i \in \mathcal{I}$. Let $\mathcal{J}(P)$ the set of all poset ideals of P. For any $\mathcal{I} \in \mathcal{J}(P)$, define

$$u_{\mathcal{I}} = (\prod_{p_i \in \mathcal{I}} x_i)(\prod_{p_i \in P \setminus \mathcal{I}} y_i).$$

The Hibi ideal (associated to P) is the monomial ideal:

$$H_P \;=\; (u_{\mathcal{I}} \;:\; \mathcal{I} \text{ is a poset ideal of } P) \;\subset\; S = K[x_i, y_i : p_i \in P].$$

The case of Hibi ideals

Theorem (F, 2023)

Let H_P be an Hibi ideal. Then:

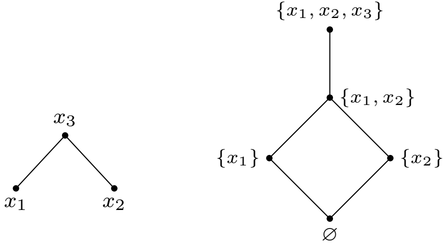
1. $\text{Ass}(H_P^k) = \text{Ass}^\infty(H_P) = \{(x_i, y_j) : p_i \preceq p_j\}$ for all $k \geq 1$.
2. For all $k \geq 1$:

$$\mathsf{v}_{(x_i,y_j)}(H_P^k) = \begin{cases} |P|k - 1 & \text{if } i = j, \\ |P|k + |\{p_\ell \in P : p_i \prec p_\ell \prec p_j\}| & \text{if } i < j. \end{cases}$$

3. $\mathsf{v}(H_P^k) = |P|k - 1$ for all $k \geq 1$.

An Example

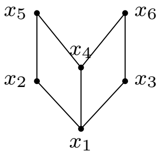
Let $(P, \succeq)$ with $P = X = \{x_1, x_2, x_3\}$, $x_3 \succ x_1$ and $x_3 \succ x_2$.



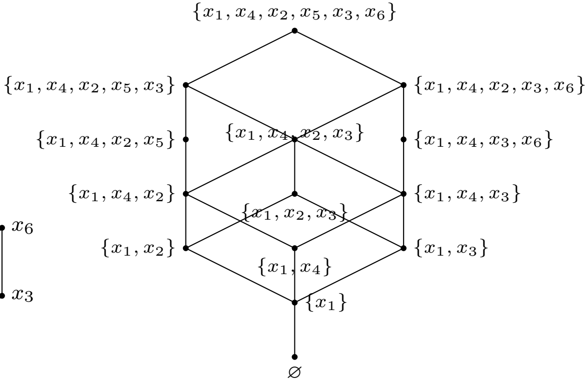
$(P, \succeq)$ $\mathcal{J}(P)$

Then, $\mathrm{v}_{(x_i,y_j)}(H_P^k) = 3k - 1$ for all $p_i = p_j$ and $\mathrm{v}_{(x_i,y_j)}(H_P^k) = 3k$ otherwise.

An example



$(P, \succeq)$



$\mathcal{J}(P)$

For instance, $\mathrm{v}_{(x_1,x_5)}(H_P^k) = 6k + 2$ for all $k \geq 1$.

Further work

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Further work

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Thanks for the attention!