On factorization invariants of ideal extensions of free commutative monoids

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C, P. A. García-Sánchez, D. Llena, Ideal extensions of free commutative monoids, *arXiv:2311.06901*, 2023.

This work is dedicated to the memory of N. Baeth

Ideal extensions of free commutative monoids

Let $\mathbb{N}$ be the additive monoid of non-negative integers.

For $I \subseteq \mathbb{N}$ non-empty, denote:

$$\mathbb{N}^{(I)} = \{(n_i)_{i\in I} \in \mathbb{N}^I \mid n_i = 0 \text{ but for finitely many } i \in I\} = \bigoplus_{i\in I} \mathbb{N}\mathbf{e}_i$$

$\mathbb{N}^{(I)}$ is the free monoid on the set $\{\mathbf{e}_i\}_{i\in I}$.

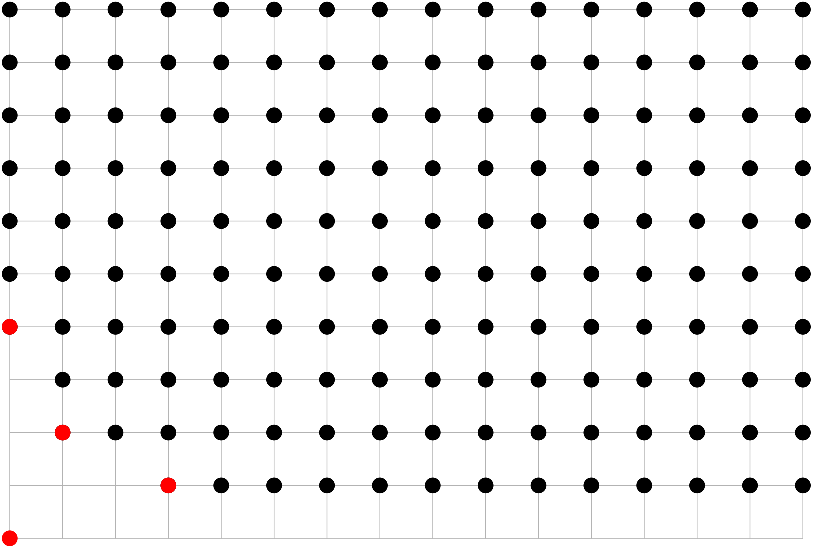
A subset E of $\mathbb{N}^{(I)}$ is an ideal if $E + \mathbb{N}^{(I)} \subseteq E$

A submonoid S of $\mathbb{N}^{(I)}$ is an ideal extension of $\mathbb{N}^{(I)}$ if

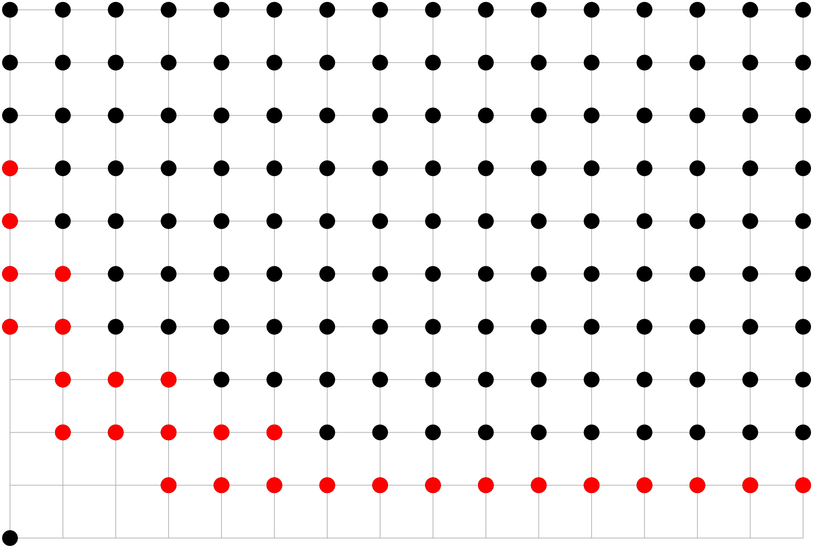
$$S = E \cup \{\mathbf{0}\} \text{ for some ideal } E \text{ of } \mathbb{N}^{(I)}.$$

- S is a commutative, cancellative and reduced.
- S is atomic with set of atoms $\mathcal{A}(S) = S^* \setminus (S^* + S^*)$.

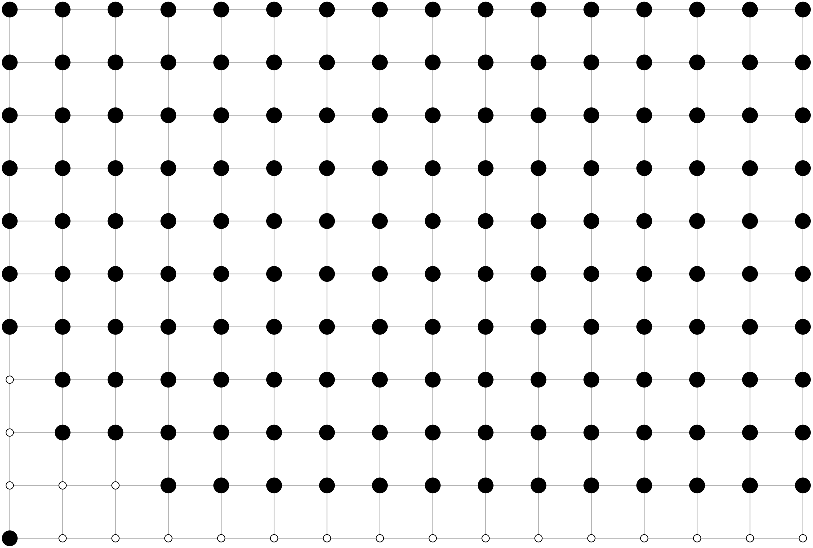
$E = \{(0,4),(1,2),(3,1)\} + \mathbb{N}^2; \quad S = \{0\} \cup E$



$\mathcal{A}(S) = S^* \setminus (S^* + S^*)$

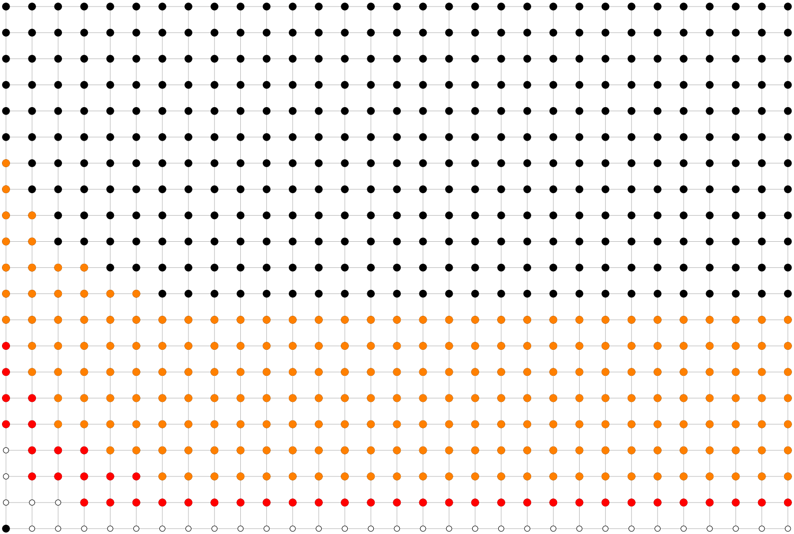


$\mathcal{H}(S) = \mathbb{N}^2 \setminus S$



(1) $2\mathcal{H}(S) \subseteq \mathcal{H}(S) \cup \mathcal{A}(S) \cup 2\mathcal{A}(S)$

(2) $\mathcal{H}(S) + \mathcal{A}(S) \subseteq \mathcal{A}(S) \cup 2\mathcal{A}(S)$



Gap absorbing monoids

Let S be a submonid of $\mathbb{N}^{(I)}$. The set of gaps of S is:

$$\mathcal{H}(S) = \mathbb{N}^{(I)} \setminus S$$

We say that S is a gap absorbing monoid if

1. $2\mathcal{H}(S) \subseteq \mathcal{H}(S) \cup \mathcal{A}(S) \cup 2\mathcal{A}(S)$, and
2. $\mathcal{H}(S) + \mathcal{A}(S) \subseteq \mathcal{A}(S) \cup 2\mathcal{A}(S)$.

Proposition

Let $S \subseteq \mathbb{N}^{(I)}$ be a gap absorbing monoid. Then S is an ideal extension of $\mathbb{N}^{(I)}$.

Given $\mathbf{m}, \mathbf{n} \in \mathbb{N}^{(I)}$, we write $\mathbf{m} \leq \mathbf{n}$ if $\mathbf{n} - \mathbf{m} \in \mathbb{N}^{(I)}$.

For $\mathbf{m}, \mathbf{n} \in \mathbb{N}^{(I)}$, we denote

$$[\![\mathbf{m}, \mathbf{n}]\!] = \{\mathbf{x} \in \mathbb{N}^{(I)} : \mathbf{m} \leq \mathbf{x} \leq \mathbf{n}\}.$$

Proposition

Let S be a submonoid of $\mathbb{N}^{(I)}$. The following conditions are equivalent:

1. *S is a gap absorbing monoid.*
2. *S is an ideal extension of $\mathbb{N}^{(I)}$ and for all $\mathbf{a}, \mathbf{b} \in 2\mathcal{A}(S)$, $[\![\mathbf{a}, \mathbf{b}]\!] \subseteq 2\mathcal{A}(S)$*

Gap absorbing monoids in $\mathbb{N}^2$

Every ideal extension of $\mathbb{N}$ is a gap absorbing monoid. In $\mathbb{N}^2$ we have:

Proposition

Let S be an ideal extension of $\mathbb{N}^2$. Then, for every $\mathbf{a}, \mathbf{b} \in 2\mathcal{A}(S)$ with $\mathbf{a} \leq \mathbf{b}$, we have that $[\![\mathbf{a}, \mathbf{b}]\!] \subseteq 2\mathcal{A}(S)$.

Corollary

Let S be a submonoid of $\mathbb{N}^2$. Then, S is gap absorbing if and only if S is an ideal extension of $\mathbb{N}^2$.

We have not counterexamples of ideal extensions S of $\mathbb{N}^{(I)}$, with $3 \leq |I| \leq \infty$, such that S is not gap absorbing.

Conjecture

Every ideal extension of $\mathbb{N}^{(I)}$ is gap absorbing.

Betti elements

If S is any additive monoid, we have the following relation:

$$\mathbf{a} \leq_S \mathbf{b} \quad \text{if} \quad \mathbf{b} = \mathbf{a} + \mathbf{c} \text{ for some } \mathbf{c} \in S.$$

To every $\mathbf{s} \in S$, we associate a graph $\mathbf{G_s} = (V_\mathbf{s}, E_\mathbf{s})$

- $V_\mathbf{s} = \{\mathbf{a} \in \mathcal{A}(S) \mid \mathbf{a} \leq_S \mathbf{s}\}$ is the set of vertices.
- $E_\mathbf{s} = \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} + \mathbf{b} \leq_S \mathbf{s}\}$ is the set of edges.

Denote:

$$\mathrm{Betti}(S) = \{\mathbf{s} \in S \mid \mathbf{G_s} \text{ is not connected }\}$$

and its elements are called Betti elements of S.

Theorem

Let $S \subseteq \mathbb{N}^{(I)}$ be a gap absorbing monoid and let $\mathbf{s} \in S$. If $\mathbf{s}$ is a Betti element, then $\mathbf{s} \in 2\mathcal{A}(S) \cup 3\mathcal{A}(S)$.

Delta sets

If $\mathbf{s} \in S$, denote:

- $\mathsf{Z}(\mathbf{s}) = \left\{ (\lambda_{\mathbf{a}})_{\mathbf{a} \in \mathcal{A}(S)} \mid \sum_{\mathbf{a} \in \mathcal{A}(S)} \lambda_{\mathbf{a}} \mathbf{a} = \mathbf{s} \right\}$ set of factorizations of **s**
- $\mathsf{L}(\mathbf{s}) = \left\{ \sum_{\mathbf{a} \in \mathcal{A}(S)} \lambda_{\mathbf{a}} \mid (\lambda_{\mathbf{a}})_{\mathbf{a} \in \mathcal{A}(S)} \in \mathsf{Z}(\mathbf{s}) \right\}$ set of lengths of **s**

If $\mathsf{L}(\mathbf{s}) = \{l_1 < l_2 < \cdots < l_r\}$.

- $\Delta(\mathbf{s}) = \{l_2 - l_1, \ldots, l_r - l_{r-1}\}$ is the delta set of **s**
- $\Delta(S) = \bigcup_{\mathbf{s} \in S} \Delta(\mathbf{s})$ is the delta set of S

S. T. Chapman, P. A. García-Sánchez, D. Llena, A. Malyshev, D. Steinberg, On the Delta set and the Betti elements of a BF-monoid, Arab. J. Math **1** (2012), 53–61.

Theorem

If S is a BF-monoid then:

$$\max \Delta(S) = \max\{\max \Delta(\mathbf{b}) \mid \mathbf{b} \in \mathrm{Betti}(S)\}$$

N. Baeth, Complement-Finite Ideals. In: Chabert, JL., Fontana, M., Frisch, S., Glaz, S., Johnson, K. (eds) Algebraic, Number Theoretic, and Topological Aspects of Ring Theory. Springer, Cham, (2023).

If S is an ideal extension of $\mathbb{N}^d$ for some positive integer d, such that $\mathbb{N}^d \setminus S$ is finite, then S is called a complement-finite ideal.

N. Baeth conjectured that if S is a complement finite ideal of $\mathbb{N}^d$ then $\Delta(S) = \{1\}$.

Baeth conjecture on delta set holds for gap absorbing monoids:

Theorem

Let S be a gap absorbing monoid. Then, $\mathsf{L}(\mathbf{s})$ is an interval for every $\mathbf{s} \in S$. Equivalently, $\Delta(S) = \{1\}$.

Catenary degree

Given $\mathbf{u} = (\lambda_{\mathbf{a}})_{\mathbf{a} \in \mathcal{A}(S)}$ and $\mathbf{v} = (\mu_{\mathbf{a}})_{\mathbf{a} \in \mathcal{A}(S)}$, denote:

$$\mathbf{u} \wedge \mathbf{v} = (\min(\lambda_{\mathbf{a}}, \mu_{\mathbf{a}}))_{\mathbf{a} \in \mathcal{A}(S)}$$

Define the distance between $\mathbf{u}$ and $\mathbf{v}$ as

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = \max\{|\mathbf{u} - (\mathbf{u} \wedge \mathbf{v})|, |\mathbf{v} - (\mathbf{u} \wedge \mathbf{v})|\}.$$

If $\mathbf{u}, \mathbf{v} \in \mathrm{Z}(\mathbf{s})$, for $\mathbf{s} \in S$, then an N-chain joining $\mathbf{u}$ and $\mathbf{v}$ is a sequence $\mathbf{u}_1, \ldots, \mathbf{u_n} \in \mathrm{Z}(\mathbf{s})$ such that $\mathbf{u}_1 = \mathbf{u}$, $\mathbf{u}_n = \mathbf{v}$, and $\mathrm{d}(\mathbf{u}_i, \mathbf{u}_{i+1}) \leq N$ for all $i \in \{1, \ldots, n-1\}$.

The catenary degree of $\mathbf{s}$, $\mathrm{c}(\mathbf{s})$, is the minimum positive integer N such that for any two factorizations of $\mathbf{s}$ there exists an N-chain joining them.

The catenary degree of S is defined as

$$\mathrm{c}(S) = \sup\{\mathrm{c}(\mathbf{s}) : \mathbf{s} \in S\}.$$

S.T. Chapman, P. A. García-Sánchez, D. Llena, V. Ponomarenko, J.C. Rosales, The catenary and tame degree in finitely generated commutative cancellative monoids. Manuscripta Math. 120, 253–264 (2006)

Proposition

If S is not half factorial, then $\mathsf{c}(S) = \sup\{\mathsf{c}(\mathbf{b}) : \mathbf{b} \in \mathrm{Betti}(S)\}$.

For gap absorbing monoid we can prove:

Theorem

Let S be a gap absorbing monoid. Then $\mathsf{c}(S) \leq 4$.

Corollary

Let S be a gap absorbing monoid. If $\mathrm{Betti}(S) \subseteq 2\mathcal{A}(S)$, *then* $\mathsf{c}(S) \leq 3$.

In particular, if S is a gap absorbing monoid in $\mathbb{N}^2$ then $\mathsf{c}(S) \leq 3$.

ω-primality

The ω-primality of $\mathbf{s}$, denoted $\omega(\mathbf{s})$, is the least positive integer N such that:

$$\text{whenever } \mathbf{s} \leq_S \mathbf{s}_1 + \cdots + \mathbf{s}_n \text{ for some } \mathbf{s}_1, \ldots, \mathbf{s}_n \in S$$

$$\Downarrow$$

$$\mathbf{s} \leq_S \textstyle\sum_{j \in J} \mathbf{s}_j \text{ for some } J \subseteq \{1, \ldots, n\} \text{ with } |J| \leq N.$$

The ω-primality of S is defined as

$$\omega(S) = \sup\{\omega(\mathbf{a}) : \mathbf{a} \in \mathcal{A}(S)\}.$$

ω-primality of ideal extensions.

Theorem

Let S be an ideal extension of $\mathbb{N}^{(I)}$. For every $\mathbf{a} \in \mathcal{A}(S)$, $\omega(\mathbf{a}) \leq \|\mathbf{a}\|_1 + 1$. In particular,

$$\omega(S) \leq 1 + \sup_{\mathbf{a} \in \mathcal{A}(S)} \|\mathbf{a}\|_1.$$

Proposition

Let S be an extension ideal of $\mathbb{N}^{(I)}$. Let $\mathbf{a} \in \mathcal{A}(S)$ and suppose there exists $\mathbf{b} \in S^*$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. Then $\omega(\mathbf{a}) \geq \|\mathbf{a}\|_1$.

Proposition

Let S be an ideal extension of $\mathbb{N}^k$ for some positive integer k. Then, $\omega(S) < \infty$ if and only if $|\mathcal{H}(S)| < \infty$.

Notable examples

Let $J \subseteq I \subseteq \mathbb{N}$, for $\mathbf{x} \in \mathbb{N}^{(I)}$ we define $|\mathbf{x}|_J = \sum_{J \in I} x_i$.

Let $T \subseteq \mathbb{N}$ be a numerical semigroup, define the following set:

$$S_J^I(T) = \{\mathbf{x} \in \mathbb{N}^{(I)} : |\mathbf{x}|_J \in T \setminus \{0\}\} \cup \{\mathbf{0}\}.$$

$S_J^I(T)$ is a submonoid of $\mathbb{N}^{(I)}$, that we call backslash monoid.

Proposition

Let T be a numerical semigroup. Then, $S_J^I(T)$ is a gap absorbing semigroup if and only if $T = \mathbb{N} \setminus \{1, \ldots, n-1\}$ for some $n \in \mathbb{N}$.

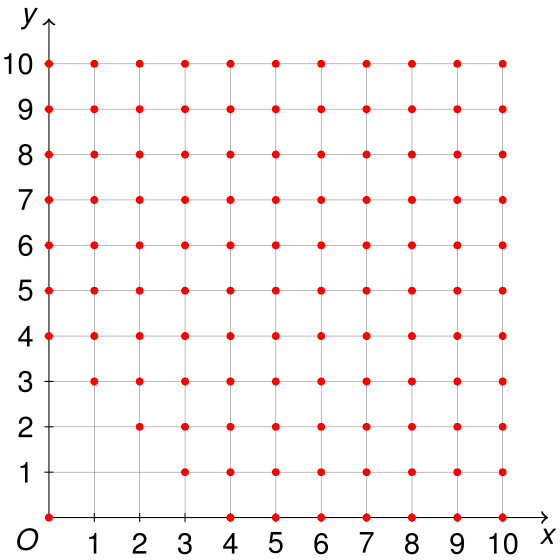


Figure: $S_J^I(T)$ with $J = I = \{1, 2\}$ and $T = \mathbb{N} \setminus \{1, 2, 3\}$

Proposition

Suppose $T = \mathbb{N} \setminus \{1, \ldots, n-1\}$ *with* $n \geq 2$ *and* $\emptyset \neq J \subseteq I$*. The following holds:*

1. $\mathrm{Betti}(S^I_J(T)) \subseteq 2\mathcal{A}(S^I_J(T))$.
2. $\mathrm{c}(S^I_J(T)) = 3$
3. *If* $|I| > 1$ *then:*
    - $\omega(S^I_I(T)) = 2n - 1$*,and*
    - $\omega(S^I_J(T)) = \infty$ *for any proper subset* J *of* I*.*

In particular, in $\mathbb{N}^{(I)}$ with $|I| > 1$ it is possible to obtain gap absorbing monoids with ω-primality as large as desired.

Open questions

- Is every ideal extension gap absorbing?
- For every ideal extension, is the minimal length of a Betti element at most two?
- Is the catenary degree of an ideal extension at most three?
- For the ω-primality, if S is an ideal extension we know that $\omega(S)$ is upper bounded by the supremum of 1-norms of its atoms plus one. We have not found any example where this upper bound is attained.

Thank you for your attention