Discrete valuation overrings of a Noetherian ring

Gyu Whan Chang

Incheon National University

July 24, 2024

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Definition and Notation, I

1. R is a commutative ring with identity.
2. $T(R)$ is the total quotient ring of R, so if R is an integral domain, then $T(R)$ is the quotient field of R.
3. An overring of R is a subring of $T(R)$ containing R.
4. $\dim R$ denotes the (Krull) dimension of R.
5. $\mathrm{ht} P = \dim(R_P)$ for a prime ideal P of R.
6. An integral domain R is a valuation ring if either $x \in R$ or $x^{-1} \in R$ for all $0 \neq x \in T(R)$.
7. A finite (Krull) dimensional valuation domain V is a discrete valuation ring (DVR) if PV_P is principal for all nonzero prime ideals P of V.

Krull-Akizuki Theorem and Chevalley's result

Krull-Akizuki Theorem. Let R be a Noetherian domain in which every nonzero prime ideal is maximal, i.e., $\dim R \leq 1$, $K = T(R)$, L be a finite extension field of K and R_0 be a subring of L containing R. Then

1. R_0 is a Noetherian domain with $\dim(R_0) \leq 1$.
2. Every integrally closed subring of L containing R is a Dedekind domain.
3. The integral closures of R in K and L, respectively, are Dedekind domains.
4. If I is a nonzero ideal of R_0 then R_0/I has finite length as an R-module.

Chevalley's result. Let R be a Noetherian domain and P be a nonzero prime ideal of R. Then P is dominated by a rank-one DVR, i.e., there is a rank-one DVR V with maximal ideal $\mathbf{m}$, so that $\mathbf{m} \cap R = P$.

Definition and Notation, II

1. An element of R is regular if it is not a zero divisor.
2. An ideal of R is regular if it contains a regular element.
3. R is an r-Noetherian ring if every regular (prime) ideal of R is finitely generated.
4. R is a Dedekind ring if each regular ideal of R is invertible.
5. Let P be a prime ideal of R. Then reg-ht$P =$ $\sup\{n \mid P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n = P$ of regular prime ideals of $R\}$.
6. reg-dim$(R) = \sup\{$reg-ht$Q \mid Q$ is a regular prime ideal of $R\}$.

• reg-ht$P \leq$ ht$P$, reg-dim$(R) \leq \dim(R)$, and equalities hold if $\dim(T(R)) = 0$ (e.g., R is an integral domain).

• reg-ht$P = 0$ if and only if P is not regular; reg-dim$(R) = 0$ if and only if $R = T(R)$; and reg-dim$(R) = 1$ if and only if $R \neq T(R)$ and each regular prime ideal of R is a maximal ideal.

Noetherian ring of reg-dim$R \leq 1$, I

Lemma (R. Matsuda; 1985)

R is a Dedekind ring if and only if R is an integrally closed r-Noetherian ring with reg-dim$(R) \leq 1$.

Theorem (Chang; 1999)

Let R be a Noetherian ring with reg-dim$(R) = 1$.

1. *If T is an overring of R, then T is an r-Noetherian ring with reg-dim$(T) \leq 1$.*
2. *Every integrally closed overring of R is a Dedekind ring.*

Example

For a given integer $n \geq 1$, there is a Noetherian ring R such that dim$R = n \geq 1 =$ reg-dim$R$ and the integral closure of R is not Noetherian.

Noetherian ring of reg-dim$R \leq 1$, II

Proof of Example.

1. Let $X_1, \ldots, X_n$ be indeterminates over $\mathbb{Q}$, $\mathbb{Q}[X_1, \ldots, X_n]$ be the polynomial ring over $\mathbb{Q}$, $(X_1, \ldots, X_n)$ be the maximal ideal of $\mathbb{Q}[X_1, \ldots, X_n]$, $D = \mathbb{Q}[X_1, \ldots, X_n]_{(X_1, \ldots, X_n)}$, and $M = (X_1, \ldots, X_n)D$. Then D is a local Noetherian domain with maximal ideal M and $\dim(D) = n$.
2. Next, let $R_1 = D(+)D/M$ be the idealization of D by D/M. Then R_1 is an n-dimensional Noetherian ring and $T(R) = R$.
3. Finally, let R_2 be a Noetherian ring with reg-dim$(R_2) = \dim(R_2) = 1$ and the integral closure of R_2 is not Noetherian, e.g., $R_2 = \mathbb{Q}[X_1, X_2]_{(X_1, X_2)}/X_1^2\mathbb{Q}[X_1, X_2]_{(X_1, X_2)}$.
4. Now, let $R = R_1 \times R_2$. Then R is a Noetherian ring, reg-dim$(R) = 1 \leq n = \dim(R)$, the integral closure R' of R is an r-Noetherian ring, but R' is not a Noetherian ring.

□

Simple overring of a Noetherian ring

Lemma (Chang; 2024)

Let R be a Neotherian ring and $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n$ be a chain of *regular prime ideals* *of R with $htP_1 \geq 2$. Then there exist some $a, b \in R$ with $a \in reg(R)$, so that there exists a chain of prime ideals $Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_n$ in $R[\frac{b}{a}]$ which satisfies the following properties:*

1. $Q_i \cap R = P_i$ *for* $i = 1, \ldots, n$,
2. $htQ_1 = htP_1 - 1$,
3. $ht(Q_i/Q_{i-1}) = ht(P_i/P_{i-1})$ *for* $i = 2, \ldots, n$,
4. $Q_i = P_iR[\frac{b}{a}]$, $R[\frac{b}{a}]/Q_i \cong (R/P_i)[X]$ *for an indeterminate* X *over* R/P_i *and* $dim(R[\frac{b}{a}]/Q_i) = dim(R/P_i) + 1$ *for* $i = 1, \ldots, n$.

Simple overring of a Noetherian ring

Proof.

Step 1. For $P = P_1$, let $\mathrm{ht}P = k$ and $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_k = P$ be a chain of regular prime ideals of R. Choose $a \in \mathrm{reg}(A_1)$ and $S = \{P' \in \mathrm{Spec}(R) \mid P' \text{ is minimal over } aR\}$. Then S is finite, each prime ideal in S has height one, and $A_1 \in S$, because R is a Noetherian ring; so one can choose $b \in A_2 \setminus \bigcup_{P' \in S} P'$.

Step 2. Let $\varphi : R[X] \to R[\frac{b}{a}]$ be a ring homomorphism defined by $\varphi(f(X)) = f(\frac{b}{a})$. Then φ is onto, so if we let $\ker(\varphi) = A$, then $R[X]/A = R[\frac{b}{a}]$. Since a is regular, by division algorithm, one can easily show that $A = (aX - b)T(R)[X] \cap R[X]$ and if $f \in A$, then $a^{\deg(f)}f = (aX - b)g$ for some $g \in R[X]$. Note that A is a finitely generated ideal of $R[X]$, so $a^mA \subseteq (aX - b)R[X]$ for some integer $m \geq 1$. □

Simple overring of a Noetherian ring

Proof.

Step 3. Let Q be a prime ideal of $R[X]$ that is minimal over $(aX-b)R[X]$ and is contained in $A_2[X]$. Then $\mathrm{ht}Q = \text{reg-ht}Q = 1$ by the principal ideal theorem, and hence $(Q\cap R)[X] \subsetneq Q \subsetneq A_2[X]$ and $a \notin Q\cap R$. Note also that $a^m A \subseteq Q$, hence $A \subseteq Q$. Thus,

$$Q/A \subsetneq A_2[X]/A \subsetneq \cdots \subsetneq A_k[X]/A$$

$$= P[X]/A \subsetneq P_2[X]/A \subsetneq \cdots \subsetneq P_n/A \subsetneq R[X]/A$$

is a chain of prime ideals of a Noetherian ring $R[X]/A$.

Step 4. Finally, let $\psi : R[X]/A \to R[\frac{b}{a}]$ be the isomorphism and $Q_i = \psi(P_i[X]/A])$. Then $Q_1 \subsetneq \cdots \subsetneq Q_n$ is the desired chain of prime ideals in $R[\frac{b}{a}]$. □

Application, I

Theorem

Let R be a Noetherian ring and P be a regular prime ideal of R with $htP = n$. Then there exists a finitely generated R-algebra S with a regular prime ideal Q that has the following properties.

1. *S is an overring of R.*
2. *$Q \cap R = P$, $Q = PS$ and $htQ = 1$.*
3. *$S/Q \cong (R/P)[X_1, \ldots, X_{n-1}]$ and $dim(S/Q) = dim(R/P) + n - 1$, where $X_1, \ldots, X_{n-1}$ are indeterminates over R/P.*

Proof.

This follows directly from the previous lemma and induction on ht$P$. □

Applications, II

Corollary

Let R be a Noetherian ring, P be a prime ideal of R with $htP = n$, and $R[X]$ be the polynomial ring over R. Then there exists a finitely generated R-algebra S with a regular prime ideal Q that has the following properties.

1. $R[X] \subseteq S \subseteq T(R[X])$.
2. $Q \cap R = P$, $Q = PS + XS$, *and* $htQ = 1$.
3. $S/Q \cong (R/P)[X_1, \ldots, X_n]$ *and* $\dim(S/Q) = \dim(R/P) + n$ *and* $X_1, \ldots, X_n$ *are indeterminates over* R/P.

Proof.

Let $P' = P[X] + XR[X]$. Then P' is a regular prime ideal of $R[X]$, $\mathrm{ht}P' = n + 1$, and $P' \cap R = P$. Hence, the result is an immediate consequence of the previous theorem. □

A Marot ring

- A ring R is a Marot ring if each regular ideal of R is generated by a set of regular elements.
- The class of Marot rings includes Noetherian rings, overrings of a Marot ring, and polynomial rings. Hence, every overring of a Noetherian ring is a Marot ring.
- A Marot ring R is a valuation ring if either $x \in R$ or $x^{-1} \in R$ for all regular elements $x \in T(R)$.
- For a prime ideal P of a Marot ring R, $R_{(P)} = R_S$ for $S = \{x \in R \setminus P \mid x \text{ is regular}\}$.
- A finite reg-dimensional Marot ring V is a discrete valuation ring (DVR) if $PV_{(P)}$ is principal for all regular prime ideals P of R.

Extension of Chevalley's result to Noetherian rings, I

Theorem

Let R be a Noetherian ring and P be a regular prime ideal of R. Then there is a rank one discrete valuation overring V of R with (regular) maximal ideal M, so that $M \cap R = P$.

Proof.

By the previous theorem, we may assume that ht$P = 1$. Hence, reg-dim$(R_{(P)}) = 1$ and the integral closure $\bar{R}$ of $R_{(P)}$ is a Dedekind ring. Let Q be a regular prime ideal of $\bar{R}$, $V = \bar{R}_{(Q)}$, and $M = Q\bar{R}_{(Q)}$. Then V is a rank one discrete valuation overring of R with maximal ideal M such that $M \cap R = P$. □

Extension of Chevalley's result to Noetherian rings, II

Corollary

Let R be a Noetherian ring and P be a prime ideal of R. Then there is a rank one discrete valuation overring W of $R[X]$ with regular maximal ideal M, so that $M \cap R = P$.

Proof.

Let $Q = P[X] + XR[X]$. Then Q is a regular prime ideal of a Noetherian ring $R[X]$ such that $Q \cap R = P$. Hence, by Corollary 7, there is a rank one discrete valuation overring W of $R[X]$ with regular maximal ideal M, so that $M \cap R[X] = Q$, and hence $M \cap R = P$. □

Discrete valuation overrings

Theorem

Let R be a Noetherian ring, $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n$ be a saturated chain of regular prime ideals of R, so $htP_1 = 1$, and V be a valuation overring of R with a chain of prime ideals $\{Q_\alpha \mid \alpha \in I\}$ such that $\bigcup_\alpha Q_\alpha$ is a maximal ideal of V and $\{Q_\alpha \cap R \mid \alpha \in I\} = \{P_1, \dots, P_n\}$. Then V is a DVR with the following properties:

1. $\{Q_\alpha \mid \alpha \in I\} = \{Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_n\}$ *and*
2. *reg-dim*$(V) = n$.

The next example shows that if the chain of the theorem is not saturated, then the theorem is not true in general.

Example of a nondiscrete valuation overring

Example

Let K be a field, G be the subgroup of the additive group of real numbers generated by e and π, and $\{X, Y\}$ be a set of indeterminates over K. Define a function v from $K[X, Y]$ into $G \cup \{\infty\}$ as follows:

$$v(0) = \infty, v(m) = k_1 e + k_2 \pi, \text{ and } v(g) = \inf\{v(m_i)\}_{i=1}^{n},$$

where $m = aX^{k_1} Y^{k_2}$ is a nonzero monomial, and $g = m_1 + \cdots + m_n$ is the unique representation of $g \in K[X, Y]$ as a sum of distinct nonzero monomials. Then v has a unique extension to a valuation v' on $K(X, Y)$, and the value group of v' is G.

Let (V, M) be the valuation ring of v'. Then V is a rank one non-discrete valuation overring of $K[X, Y]$, $G =< e, \pi >\subsetneq (\mathbb{R}, +)$, $M \cap K[X, Y] = (X, Y)$, and $(0) \subsetneq (X, Y)$ is not saturated.

Applications, III

Let $\kappa(P) = T(R/P)$ for a prime ideal P of R and $\text{tr.deg}_F K$ denote the transcendental degree of K over F for an extension $F \subseteq K$ of fields. The next corollary is a strong version of Chevalley' result.

Corollary

Let R be a Noetherian ring and P be a regular prime ideal of R with $htP = n$. Then there is a rank n discrete valuation overring V of R with the chain of regular prime ideals, say, $Q_1 \subsetneq \cdots \subsetneq Q_n$, so that, for $i = 1, \ldots, n$,

1. $Q_i \cap R = P$,
2. $htQ_i = i$,
3. $\text{tr.deg}_{\kappa(P)} \kappa(Q_i) = n - i$, *and*
4. $V_{(Q_i)}$ *is a rank i discrete valuation overring of R all of whose regular prime ideals lie over P.*

Applications, IV

Corollary

Let R be a Noetherian ring, P be a prime ideal of R with $htP = n$, and $R[X]$ be the polynomial ring over R. Then there is a rank $n+1$ discrete valuation overring W of $R[X]$ with the chain of regular prime ideals, say, $Q_1 \subsetneq \cdots \subsetneq Q_n \subsetneq Q_{n+1}$, so that, for $i = 1, \ldots, n, n+1$,

1. $Q_i \cap R = P$,
2. $htQ_i = i$,
3. $tr.deg_{\kappa(P)} \kappa(Q_i) = n - i + 1$, *and*
4. $W_{(Q_i)}$ *is a rank i discrete valuation overring of R all of whose regular prime ideals lie over P.*

Moreover, if $n \geq 1$, then there are infinitely many such DVRs of rank $n+1$.

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Thank you for your attention