Multiplicative lattices, their primes and spectral spaces

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Joint papers with A. Facchini and G. Janelidze

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Ingredients

Definition

A topological space X is *spectral* if

- X is quasi-compact;
- X has a basis $\mathcal{B}$ of quasi-compact open sets such that $A, B \in \mathcal{B} \Longrightarrow A \cap B \in \mathcal{B}$;
- X is sober.

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Definition

Let R be a commutative ring with 1. Then the Zariski topology on $\mathrm{Spec}(R) := \{\text{prime ideals of } R\}$ is given by the closed sets of the type

$$V(\mathfrak{i}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{i}\} \qquad (\mathfrak{i} \trianglelefteq R)$$

Theorem (Hochster, 1969)

X is spectral iff $X \cong \mathrm{Spec}(R)$.

Ubiquity of spectra

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- R commutative ring with 1 $\mapsto$ $\mathrm{Spec}(R)$.
- L bounded distributive lattice $\mapsto$ $\mathrm{Spec}(L)$.
- S commutative semiring with 1 $\mapsto$ $\mathrm{Spec}(S)$.
- M commutative monoid $\mapsto$ $\mathrm{Spec}(M)$.
- G group $\mapsto$ $\mathrm{Spec}(G)$.
- R noncommutative ring with 1 $\mapsto$ $\mathrm{Spec}(R)$.
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1. A. Facchini, C.A. Finocchiaro, G. Janelidze, Abstractly constructed prime spectra. *Algebra Universalis* **83** (2022), no. 1, Paper No. 8, 38 pp.
2. A. Facchini, C.A. Finocchiaro, Multiplicative lattices: maximal implies prime and related questions, *J. Algebra Appl.*, to appear.

Multiplicative lattices

A *multiplicative lattice* is a complete lattice L endowed with an operation

$$\cdot : L \times L \to L, \quad (x,y) \mapsto x \cdot y$$

satisfying $x \cdot y \leq x \wedge y$, for every $x, y \in L$.

- W. Krull, Axiomatische Begründung der allgemeinen Idealtheorie. Sitzber. d. phys.-med. Soc. Erlangen **56**, 47-63 (1924).
- M. Ward, R. Dilworth, Residuated lattices. *Trans. Amer. Math. Soc.* **45** (1939), no. 3, 335-354.
- D. D. Anderson, E. W. Johnson, Abstract ideal theory from Krull to the present. In: Anderson, D.D., Papick, I.J. (eds.) Ideal Theoretic Methods in Commutative Algebra (Columbia, MO, 1999). Lecture Notes in Pure and Applied Mathematics, vol. 220. Dekker, New York, pp. 27-47 (2001)

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Examples

- Let R be a ring and let $L := \mathcal{I}(R) = \{\text{two-sided ideals of } R\}$. $\cdot$ is the usual multiplication of ideals. Notice that $\mathfrak{i} \cdot \mathfrak{j} \subseteq \mathfrak{i} \cap \mathfrak{j}$, for every $\mathfrak{i}, \mathfrak{j} \in L$.
- Let L be a complete lattice. Then we can take $\cdot = \wedge$.
- Let G be a group and let $L := \{\text{normal subgroups of } G\}$. Then $H \cdot K := [H,K]$.

Warning: $\cdot$ can fail to be commutative!!

Example

$\cdot$ = multiplication of ideals of noncommutative rings.

Warning: $\cdot$ can fail to be associative!!

Example

Let $G := \mathbf{S}_3$, $L := \{\text{normal subgroups of } G\} = \{G, \mathbf{A}_3, \{1\}\}$, $A \cdot B := [A, B]$, for $A, B \in L$. Then

$$[[G,G],\mathbf{A}_3] = [\mathbf{A}_3,\mathbf{A}_3] = \{1\}$$

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IDEA: if R is a ring and $\mathfrak{p} \in \mathcal{I}(R) \setminus \{R\}$, then $\mathfrak{p}$ is prime iff whenever $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}(R)$, then $\mathfrak{a} \cdot \mathfrak{b} \subseteq \mathfrak{p}$ implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.

Definition

Let L be a multiplicative lattice, let 1 (resp., 0) be the maximum (resp., the minimum) of L. An element $p \in L$ is *prime* if

(1) $p \neq 1$ and

(2) for every $x, y \in L$, $x \cdot y \leq p$ implies $x \leq p$ or $y \leq p$.

The set $\mathrm{Spec}(L) := \{\text{prime elements of } L\}$ is *the prime spectrum of L*.

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- E. Schenkman, The similarity between the properties of ideals in commutative rings and the properties of normal subgroups of groups. *Proc. Am. Math. Soc.* **9**, 375-381 (1958)
- A. Facchini, F. de Giovanni and M. Trombetti, Spectra of groups, *Algebras and Representation Theory* (2022).

The Zariski topology

Let L be a multiplicative lattice. For every $x \in L$, let

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For every $x, y \in L$ and $S \subseteq L$ we have:

- $\mathrm{V}(1) = \emptyset$ and $\mathrm{V}(0) = \mathrm{Spec}(L)$;
- $\mathrm{V}(x) \cup \mathrm{V}(y) = \mathrm{V}(x \cdot y)$;
- $\bigcap_{s \in S} \mathrm{V}(s) = \mathrm{V}(\bigvee S)$.

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Proposition

$\mathrm{Spec}(L)$ is sober.

Definition

Let L be a multiplicative lattice.

- An element $x \in L$ is *a radical element* if $x = \bigwedge S$, for some $S \subseteq \mathrm{Spec}(L)$. Let
$$\mathrm{Rad}(L) := \{\text{radical elements of } L\}.$$
- For every $x \in L$, let $\sqrt{x} := \bigwedge \mathrm{V}(x)$ be *the radical of* x.

- The map $\sqrt{-} : L \to L,\ x \mapsto \sqrt{x}$, is a closure operator.
- For every $x, y \in L$, $\mathrm{V}(x) = \mathrm{V}(\sqrt{x})$ and $\sqrt{x} \leqslant \sqrt{y} \iff \mathrm{V}(y) \subseteq \mathrm{V}(x)$.

Let X a complete lattice. Recall that an element $c \in X$ is compact if whenever $S \subseteq X$ and $c \leqslant \bigvee S$ then $c \leqslant \bigvee F$, for some finite subet $F \subseteq S$. The lattice X is said to be algebraic if every element of X is the join of compact elements of X.

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- The mapping $\operatorname{Rad}(L) \to \mathcal{O}(L)$, $x \mapsto \mathrm{D}(x)$ is an isomorphism of posets.
- In particular, $\operatorname{Rad}(L)$ is a complete lattice, but given $x, y \in \operatorname{Rad}(L)$, it can happen that $x \cdot y \notin \operatorname{Rad}(L)$.

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Proposition

Let L be a multiplicative lattice. Then the following are equivalent.

1. $\operatorname{Spec}(L)$ is spectral.
2. $\operatorname{Rad}(L)$ is algebraic, 1 is compact in $\operatorname{Rad}(L)$, and if x, y are compact in $\operatorname{Rad}(L)$, then so is $x \wedge y$.

Definitions

Let L be a multiplicative lattice.

- We say that L satisfies the *monotonicity condition* if, for every
$$x,y,z,t \in L,\ (x \leqslant y, z \leqslant t) \Rightarrow x \cdot z \leqslant y \cdot t.$$
- We say that L satisfies the *weak Kaplansky condition* if for every compact element $x \in L$, then x^2 is still compact.
- We say that L satisfies the *Kaplansky condition* if, for all compact elements $x, y \in L$, then $x \cdot y$ is still compact.
- We say that L is *m-distributive* if
$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z) \text{ and } (x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z) \qquad \forall x,y,z \in L.$$

Theorem

Let L be a multiplicative lattice with the following properties:

1. L is algebraic, m-distributive and 1 is compact;
2. L satisfies the weak Kaplansky condition;
3. if $x, y \in L$ are compact, there is a compact element $z \in L$ such that $\sqrt{x \cdot y} = \sqrt{z}$.

Then $\mathrm{Spec}(L)$ is spectral.

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Special case

Let L be a multiplicative lattice with the following properties:

1. L is algebraic, m-distributive and 1 is compact;
2. L satisfies the Kaplansky condition.

Then $\mathrm{Spec}(L)$ is spectral.

Definition (Kaplansky, 1974)

A ring R with 1 is *neo-commutative* if

$$(\mathfrak{a}, \mathfrak{b} \trianglelefteq R \text{ finitely generated}) \Rightarrow \mathfrak{a}\mathfrak{b} \text{ finitely generated}$$

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Corollary

If R is neo-commutative, then $\operatorname{Spec}(R)$ is spectral.

A prime ideal principle

Let R be a commutative ring.

Theorem (Krull, 1929)

Let $S \subseteq R$ be multiplicative. Then every maximal element of

$$\{\mathfrak{i} \in \mathcal{I}(R) \mid \mathfrak{i} \cap S = \emptyset\}$$

is prime.

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Theorem (Cohen, 1950)

R is Noetherian iff every prime ideal of R is finitely generated.

A prime ideal principle

Let L be a multiplicative lattice, $A \subseteq L$, $a, b \in L$, and let

$$(a :_l b) := \bigvee\{x \in L \mid x \cdot b \leq a\} \qquad (a :_r b) := \bigvee\{x \in L \mid b \cdot x \leq a\}.$$

- We say that L is A-generated if, for every $x \in L$, $x = \bigvee F$, for some $F \subseteq A$.
- A set $F \subseteq L$ is *left A-Oka* if $1 \in F$ and, given elements $x \in A, y \in L$,
$$(x \vee y \in F \text{ and } (y :_l x) \in F) \Rightarrow y \in F.$$
- A set $F \subseteq L$ is *A-Oka* if $1 \in F$ and, given elements $x \in A, y \in L$,
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Example

Let R be a commutative ring, $L := \mathcal{I}(R)$, $A := \{xR \mid x \in R\}$

1. $\{$finitely generated ideals of $R\}$ is A-Oka.

A prime ideal principle

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Example

Let R be a commutative ring, $L := \mathcal{I}(R)$, $A := \{xR \mid x \in R\}$

1. $\{$finitely generated ideals of $R\}$ is A-Oka.
2. If $S \subseteq R$ is multiplicative, then $\{\mathfrak{i} \in \mathcal{I}(R) \mid \mathfrak{i} \cap S \neq \emptyset\}$ is A-Ako.

A prime ideal principle

Theorem

Let L be a multiplicative lattice with the monotonicity condition and that is A-generated, for some $A \subseteq L$. Assume that a set $F \subseteq L$ satisfies one of the following conditions:

- F is a left A-Oka subset;
- F is a right A-Oka subset;
- F is an A-Oka subset and the multiplication of L is associative;
- F is an A-Ako subset.

Then every maximal element of $L \setminus F$ is prime.